An algebraic formulation of dependent type theory

Egbert Rijke

Carnegie Mellon University
erijke@andrew.cmu.edu

November 7th – 10th 2014

Goal and acknowledgements

We will do two things:

- Give a syntactic reformulation of type theory in the usual style of name-free type theory.
- Define the corresponding algebraic objects, called **E-systems**, in a category with finite limits.

For many insights in the current presentation I am grateful to

- Vladimir Voevodsky
- Richard Garner
- Steve Awodey

Desired properties of the algebraic theory

- The meta-theory should not require anything more than rules for handling inferences. In particular, natural numbers should not be required.
- Finitely many sorts. The basic ingredients are contexts, families and terms. Thus there will be 3 sorts.
- The theory is algebraic and the operations on it are homomorphisms. The wish that operations are homomorphisms tells which judgmental equalities to require.
- Finite set of rules. The theory is invariant under slicing as a consequence, rather than by convention. There will be no rule schemes.

The scope of the theory

The requirement that the theory is essentially algebraic gives immediate access to the standard tools of algebra, such as

- free algebras;
- quotients algebras.

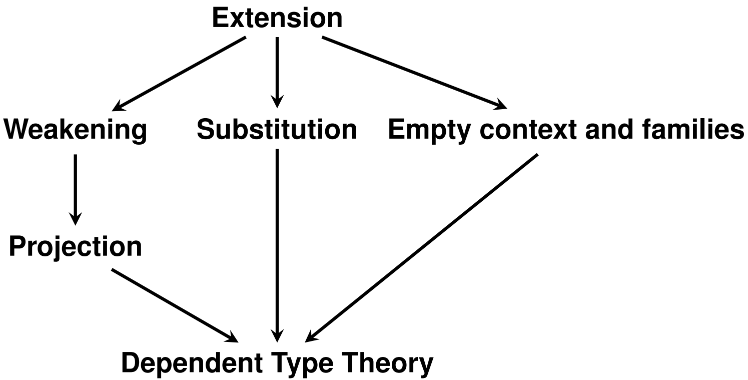
The major source of examples of E-systems are **B-systems**.

- B-systems are stratified E-systems.
- There is a full and faithful functor from C-systems to pre-B-systems with the B-systems as its image.

In a way similar to the definition of E-systems, it is possible to define **internal E-systems**. One of the aims of this research was to have a systematic treatment of internal models.

- Nothing more than plain dependent type theory is needed to express what an internal E-system is.

Overview of the theories



The theories of weakening (and projections), substitution and the empty context and families are formulated as three independent theories on top of the theory of extension.

The basic judgments

$$\vdash \Gamma \;\; \mathit{ctx} \qquad \vdash \Gamma \equiv \Delta \;\; \mathit{ctx}$$

$$\Gamma \vdash A \;\; \mathit{fam} \qquad \Gamma \vdash A \equiv B \;\; \mathit{fam}$$

$$\Gamma \vdash a : A \qquad \Gamma \vdash a \equiv b : A.$$

There are rules expressing that all three kinds of judgmental equality are convertible equivalence relations.
An example of a conversion rule:

$$\frac{\vdash \Gamma \equiv \Delta \;\; \mathit{ctx} \quad \Gamma \vdash A \;\; \mathit{fam}}{\Delta \vdash A \;\; \mathit{fam}} \qquad \frac{\vdash \Gamma \equiv \Delta \;\; \mathit{ctx} \quad \Gamma \vdash A \equiv B \;\; \mathit{fam}}{\Delta \vdash A \equiv B \;\; \mathit{fam}}$$

The fundamental structure of E-systems

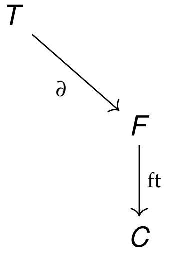
The **fundamental structure of an E-system** CFT in a category with finite limits consists of

$$\begin{array}{c} T \\ \Big\downarrow \partial \\ F \\ \Big\downarrow \mathrm{ft} \\ C \end{array}$$

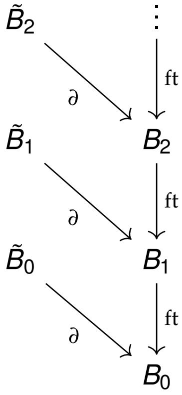
- The object C represents the sort of contexts.
- The object F represents the sort of families.
- Every family has a context, this is represented by $\mathrm{ft} : F \to C$.
- The object T represents the sort of terms.
- Every term has a family associated to it, this is represented by $\partial : T \to F$.

Comparison with B-systems

The fundamental structure of E-systems:



The underlying structure of B-systems:



Rules for extension

Introduction rules for **context extension**:

$$\frac{\Gamma \vdash A \;\; \mathit{fam}}{\vdash \Gamma.A \;\; \mathit{ctx}} \qquad \frac{\vdash \Gamma \equiv \Delta \;\; \mathit{ctx} \quad \Gamma \vdash A \equiv B \;\; \mathit{fam}}{\vdash \Gamma.A \equiv \Delta.B \;\; \mathit{ctx}}$$

Introduction rules for **family extension**:

$$\frac{\Gamma.A \vdash P \;\; \mathit{fam}}{\Gamma \vdash A.P \;\; \mathit{fam}} \qquad \frac{\Gamma \vdash A \equiv B \;\; \mathit{fam} \quad \Gamma.A \vdash P \equiv Q \;\; \mathit{fam}}{\Gamma \vdash A.P \equiv B.Q \;\; \mathit{fam}}$$

Extension is associative:

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma.A \vdash P \;\; \mathit{fam}}{\vdash (\Gamma.A).P \equiv \Gamma.(A.P) \;\; \mathit{ctx}}$$

$$\frac{\Gamma.A \vdash P \;\; \mathit{fam} \quad (\Gamma.A).P \vdash Q \;\; \mathit{fam}}{\Gamma \vdash A.(P.Q) \equiv (A.P).Q \;\; \mathit{fam}}$$

Pre-extension algebras

A **pre-extension algebra** CFT in a category with finite limits consists

- a fundamental structure CFT, and
- context extension and family extension operations

$$\mathrm{e}_0 : F \to C$$
$$\mathrm{e}_1 : F \times_{\mathrm{e}_0,\mathrm{ft}} F \to F,$$

implementing the introduction rules

$$\frac{\Gamma \vdash A \;\; \mathit{fam}}{\vdash \Gamma.A \;\; \mathit{ctx}} \qquad \frac{\Gamma.A \vdash P \;\; \mathit{fam}}{\Gamma \vdash A.P \;\; \mathit{fam}}$$

Thus, we will additionally require that we have a commuting diagram

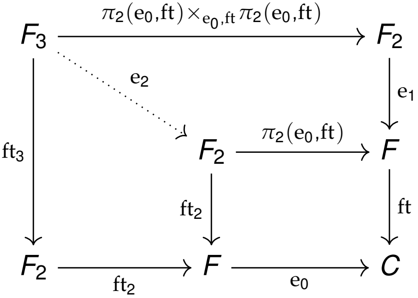
$$\begin{array}{ccc} F \times_{\mathrm{e}_0,\mathrm{ft}} F & \xrightarrow{\mathrm{e}_1} & F \\ {\scriptstyle \pi_1(\mathrm{e}_0,\mathrm{ft})}\downarrow & & \downarrow{\scriptstyle \mathrm{ft}} \\ F & \xrightarrow[\mathrm{ft}]{} & C \end{array}$$

Notation for pre-extension algebras

We introduce the following notation:

$$\begin{aligned} F_2 &:= F \times_{\mathrm{e}_0, \mathrm{ft}} F \\ \mathrm{ft}_2 &:= \pi_1(\mathrm{e}_0, \mathrm{ft}) : F_2 \to F \\ F_3 &:= F_2 \times_{\mathrm{e}_1, \mathrm{ft}_2} F_2 \\ \mathrm{ft}_3 &:= \pi_1(\mathrm{e}_1, \mathrm{ft}_2) : F_3 \to F_2. \end{aligned}$$

Then it follows that the outer square in the diagram



commutes. We define e_2 to be the unique morphism rendering the above diagram commutative.

Extension algebras

An **extension algebra** is a pre-extension algebra CFT for which the diagrams

$$
\begin{array}{ccc}
F_2 & \xrightarrow{\mathrm{e}_1} & F \\
{\scriptstyle \pi_2(\mathrm{e}_0,\mathrm{ft})}\downarrow & & \downarrow{\scriptstyle \mathrm{e}_0} \\
F & \xrightarrow[\mathrm{e}_0]{} & C
\end{array}
\qquad\qquad
\begin{array}{ccc}
F_3 & \xrightarrow{\mathrm{e}_2} & F_2 \\
{\scriptstyle \pi_2(\mathrm{e}_1,\mathrm{ft}_2)}\downarrow & & \downarrow{\scriptstyle \mathrm{e}_1} \\
F_2 & \xrightarrow[\mathrm{e}_1]{} & F
\end{array}
$$

commute.

- These diagrams implement associativity of extension.

Subgoal: develop properties of extension algebras

We need to know:

- What homomorphisms of (pre-)extension algebras are.
- That each extension algebra gives locally an extension algebra (stable under slicing).
- That the change of base of an extension is an extension algebra. Change of base allows for **'parametrized extension homomorphisms'** of which weakening and substitution are going to be examples.

Pre-extension homomorphisms

Let CFT and CFT' be pre-extension algebras. A **pre-extension homomorphism f from CFT' to CFT** is a triple (f_0, f_1, f^t) consisting of morphisms

$$\begin{array}{ccc} T' & \xrightarrow{f^t} & T \\ {\scriptstyle \partial'}\big\downarrow & & \big\downarrow{\scriptstyle \partial} \\ F' & \xrightarrow{f_1} & F \\ {\scriptstyle \mathrm{ft}'}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ft}} \\ C' & \xrightarrow[f_0]{} & C \end{array}$$

such that the indicated squares commute, for which furthermore the squares

$$\begin{array}{ccc} F' & \xrightarrow{f_1} & F \\ {\scriptstyle \mathrm{e}'_0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{e}_0} \\ C' & \xrightarrow[f_0]{} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} F'_2 & \xrightarrow{f_1 \times_{\mathrm{e}_0,\mathrm{ft}} f_1} & F_2 \\ {\scriptstyle \mathrm{e}'_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{e}_1} \\ F' & \xrightarrow[f_1]{} & F \end{array}$$

commute.

Extension homomorphisms

Definition

A pre-extension homomorphism between extension algebras is called an **extension homomorphism**.

Slicing of pre-extension algebras

Suppose that CFT is a pre-extension algebra. Then we define the pre-extension algebra $\mathbf{F}_{\mathrm{CFT}}$ to consist of the fundamental structure

$$\begin{array}{c} T_2 \\ \downarrow \partial_2 \\ F_2 \\ \downarrow \mathrm{ft}_2 \\ F \end{array}$$

where

$$T_2 := F \times_{\mathrm{e}_0, \mathrm{ft} \circ \partial} T, \qquad \partial_2 := \mathrm{e}_0^*(\partial),$$

with the extension operations

$$\mathrm{e}_1 : F_2 \to F, \qquad \mathrm{e}_2 : F_3 \to F_2.$$

Local extension algebras

Theorem

If CFT *is an extension algebra, then so is* $\mathbf{F}_{\mathrm{CFT}}$*.*

Lemma

Let CFT *be a pre-extension algebra. Then* CFT *is an extension algebra if and only if we have the extension homomorphisms* $\mathbf{e}_0 : \mathbf{F}_{\mathrm{CFT}} \to \mathrm{CFT}$ *and* $\mathbf{e}_1 : \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} \to \mathbf{F}_{\mathrm{CFT}}$ *given by*

$$\begin{CD} T_2 @>{\pi_2(\mathrm{e}_0,\mathrm{ft}\circ\partial)}>> T \\ @V{\partial_2}VV @VV{\partial}V \\ F_2 @>{\pi_2(\mathrm{e}_0,\mathrm{ft})}>> F \\ @V{\mathrm{ft}_2}VV @VV{\mathrm{ft}}V \\ F @>>{\mathrm{e}_0}> C \end{CD} \qquad \textit{and} \qquad \begin{CD} T_3 @>{\pi_2(\mathrm{e}_1,\mathrm{ft}_2\circ\partial_2)}>> T_2 \\ @V{\partial_3}VV @VV{\partial_2}V \\ F_3 @>{\pi_2(\mathrm{e}_1,\mathrm{ft}_2)}>> F_2 \\ @V{\mathrm{ft}_3}VV @VV{\mathrm{ft}_2}V \\ F_2 @>>{\mathrm{e}_1}> F \end{CD}$$

Condensed commutative diagrams

Suppose $f : \mathrm{CFT} \to \mathrm{CFT}'$ is a pre-extension homomorphism. We say that a diagram

$$\begin{array}{ccc} \mathrm{CFT} & \xrightarrow{f} & \mathrm{CFT}' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ X & \xrightarrow[g]{} & Y \end{array}$$

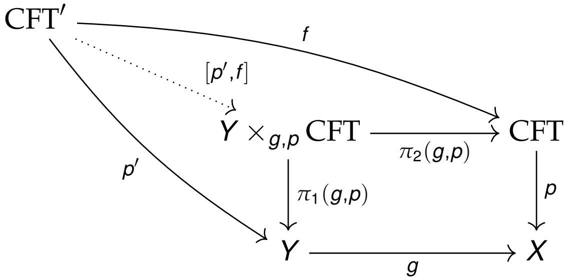
commutes if the diagram

$$\begin{array}{ccc} C & \xrightarrow{f_0} & C' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ X & \xrightarrow[g]{} & Y \end{array}$$

commutes.

Change of base of pre-extension algebras

Let CFT be a pre-extension algebra and consider $p : C \to X \leftarrow Y : g$. Then there is a pre-extension algebra $Y \times_{g,p} \mathrm{CFT}$ with projections, such that for every diagram



of which the outer square commutes, the pre-extension homomorphism $[p', f]$ exists and is unique with the property that it renders the diagram commutative.

Change of base of extension algebras

Theorem

The change of base of an extension algebra is again an extension algebra.

Theorem

Let CFT *be a pre-extension algebra. Then*

$$\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} \simeq F \times_{\mathrm{e}_0,\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}}.$$

Theorem

Let CFT *be a pre-extension algebra and consider* $p : C \to X \leftarrow Y : g$*. Then*

$$\mathbf{F}_{Y \times_{g,p} \mathrm{CFT}} \simeq Y \times_{g, p \circ \mathrm{ft}} \mathbf{F}_{\mathrm{CFT}}.$$

Overview of the theory of weakening

- In the syntax, we will introduce an operation for weakening which acts at the three levels of fundamental structures (contexts, families and terms). **When B is weakened by A, we denote this by $\langle A \rangle B$.**
- Weakening will preserve extension, so it will be implemented as an extension homomorphism.
- There will be a notion of **'Currying for weakening'**, which explains what happens when something is weakened by an extended object.
- Pre-weakening algebras will be extension algebras with a weakening operation satisfying an implementation of the currying condition. Pre-weakening homomorphisms will preserve this structure.
- Weakening will preserve itself, so weakening we will require that weakening is a pre-weakening homomorphism.

Rules for the theory of weakening

The introduction rules for weakening:

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma \vdash B \;\; \mathit{fam}}{\Gamma.A \vdash \langle A \rangle B \;\; \mathit{fam}} \qquad \frac{\Gamma \vdash A \equiv A' \;\; \mathit{fam} \quad \Gamma \vdash B \equiv B' \;\; \mathit{fam}}{\Gamma.A \vdash \langle A \rangle B \equiv \langle A' \rangle B' \;\; \mathit{fam}}$$

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma.B \vdash Q \;\; \mathit{fam}}{(\Gamma.A).\langle A \rangle B \vdash \langle A \rangle Q \;\; \mathit{fam}} \qquad \frac{\Gamma \vdash A \equiv A' \;\; \mathit{fam} \quad \Gamma.B \vdash Q \equiv Q' \;\; \mathit{fam}}{(\Gamma.A).\langle A \rangle B \vdash \langle A \rangle Q \equiv \langle A' \rangle Q' \;\; \mathit{fam}}$$

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma.B \vdash g : Q}{(\Gamma.A).\langle A \rangle B \vdash \langle A \rangle g : \langle A \rangle Q} \qquad \frac{\Gamma \vdash A \equiv A' \;\; \mathit{fam} \quad \Gamma.B \vdash g \equiv g' : Q}{(\Gamma.A).\langle A \rangle B \vdash \langle A \rangle g \equiv \langle A' \rangle g' : \langle A \rangle Q}$$

Weakening preserves extension:

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma.B \vdash Q \;\; \mathit{fam}}{\Gamma.A \vdash \langle A \rangle (B.Q) \equiv (\langle A \rangle B).\langle A \rangle Q \;\; \mathit{fam}}$$

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad (\Gamma.B).Q \vdash R \;\; \mathit{fam}}{(\Gamma.A).\langle A \rangle B \vdash \langle A \rangle (Q.R) \equiv (\langle A \rangle Q).\langle A \rangle R \;\; \mathit{fam}}$$

The weakening operation

Let CFT be an extension algebra. A **weakening operation on CFT** is an extension homomorphism

$$\mathbf{w}(\mathrm{CFT}) : F \times_{\mathrm{ft},\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}} \to \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}$$

for which the diagram

$$\begin{array}{ccc} F \times_{\mathrm{ft},\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}} & \xrightarrow{\mathbf{w}(\mathrm{CFT})} & \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} \\ & \searrow^{\pi_1(\mathrm{ft},\mathrm{ft})} & \downarrow \mathrm{ft}_2 \\ & & F \end{array}$$

commutes.

Rules for weakening: Currying

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma.A \vdash P \;\; \mathit{fam} \quad \Gamma \vdash B \;\; \mathit{fam}}{(\Gamma.A).P \vdash \langle A.P \rangle B \equiv \langle P \rangle \langle A \rangle B \;\; \mathit{fam}}$$

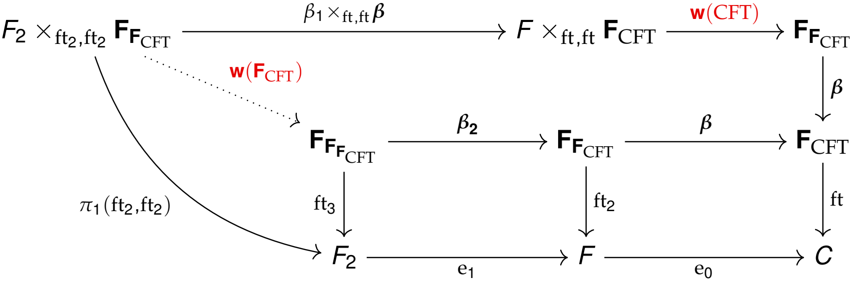
$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma.A \vdash P \;\; \mathit{fam} \quad \Gamma.B \vdash Q \;\; \mathit{fam}}{((\Gamma.A).P).\langle P \rangle \langle A \rangle B \vdash \langle A.P \rangle Q \equiv \langle P \rangle \langle A \rangle Q \;\; \mathit{fam}}$$

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma.A \vdash P \;\; \mathit{fam} \quad \Gamma.B \vdash g : Q}{((\Gamma.A).P).\langle P \rangle \langle A \rangle B \vdash \langle A.P \rangle g \equiv \langle P \rangle \langle A \rangle g : \langle P \rangle \langle A \rangle Q}$$

To express these rules algebraically, we need to define a weakening operation on $\mathbf{F}_{\mathrm{CFT}}$, provided we have one on CFT.

Weakening for the families algebra

Let CFT be an extension algebra with weakening operation $\mathbf{w}(\mathrm{CFT})$. Then $\mathbf{F}_{\mathrm{CFT}}$ has the weakening operation $\mathbf{w}(\mathbf{F}_{\mathrm{CFT}})$ which is uniquely determined by rendering the diagram



commutative.

Pre-weakening algebras

A **pre-weakening algebra** is an extension algebra CFT with a weakening operation $\mathbf{w}(\mathrm{CFT}) : F \times_{\mathrm{ft},\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}} \to \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}$ for which the diagram

$$F_2 \times_{\mathrm{ft}\circ\mathrm{ft}_2,\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}} \xrightarrow{[\pi_1(\mathrm{ft}\circ\mathrm{ft}_2,\mathrm{ft}),\mathbf{w}(\mathrm{CFT})\circ(\mathrm{ft}_2 \times_{\mathrm{ft},\mathrm{ft}} \mathrm{id}_{\mathbf{F}_{\mathrm{CFT}}})]} F_2 \times_{\mathrm{ft}_2,\mathrm{ft}_2} \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}$$

$$F_2 \times_{\mathrm{ft}_2,\mathrm{ft}_2} \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} \xrightarrow{\mathbf{w}(\mathbf{F}_{\mathrm{CFT}})} \mathbf{F}_{\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}}$$

$$F_2 \times_{\mathrm{ft}\circ\mathrm{ft}_2,\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}} \xrightarrow{[\pi_1(\mathrm{ft}\circ\mathrm{ft}_2,\mathrm{ft}),\mathbf{w}(\mathrm{CFT})\circ(\mathrm{e}_1 \times_{\mathrm{ft},\mathrm{ft}} \mathrm{id}_{\mathbf{F}_{\mathrm{CFT}}})]} \mathbf{F}_{\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}}$$

commutes (implementing currying for weakening).

Pre-weakening homomorphisms

A **pre-weakening homomorphism** between pre-weakening algebras CFT' and CFT is an extension homomorphism $f : \mathrm{CFT}' \to \mathrm{CFT}$ such that additionally the diagram

$$\begin{array}{ccc}
F' \times_{\mathrm{ft}',\mathrm{ft}'} \mathbf{F}_{\mathrm{CFT}'} & \xrightarrow{f_1 \times_{\mathrm{ft},\mathrm{ft}} \mathbf{F}_f} & F \times_{\mathrm{ft},\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}} \\
\Big\downarrow {\scriptstyle \mathbf{w}(\mathrm{CFT}')} & & \Big\downarrow {\scriptstyle \mathbf{w}(\mathrm{CFT})} \\
\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}'}} & \xrightarrow[\mathbf{F}_{\mathbf{F}_f}]{} & \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}
\end{array}$$

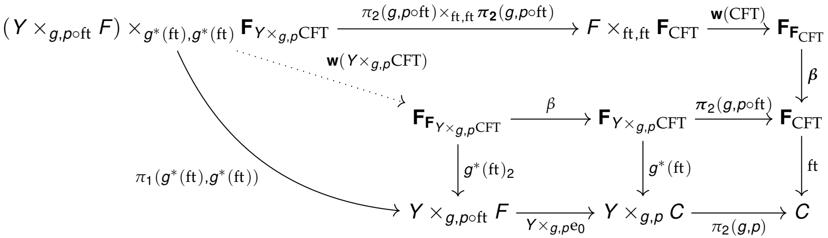
commutes.

Change of base of pre-weakening algebras

Let CFT be a pre-weakening algebra and consider $p : C \to X \leftarrow Y : g$. Then we define

$$\mathbf{w}(Y \times_{g,p} \mathrm{CFT}) : (Y \times_{g,p\circ \mathrm{ft}} F) \times_{g^*(\mathrm{ft}),g^*(\mathrm{ft})} \mathsf{F}_{Y\times_{g,p}\mathrm{CFT}} \to \mathsf{F}_{\mathsf{F}_{Y\times_{g,p}\mathrm{CFT}}}$$

to be the unique extension homomorphism rendering the diagram



commutative.

Properties of pre-weakening algebras

Theorem

If CFT is a pre-weakening algebra and $p : C \to X \leftarrow Y : g$, then $Y \times_{g,p} \mathrm{CFT}$ is also a pre-weakening algebra.

Theorem

If CFT is a pre-weakening algebra, then so is $\mathbf{F}_{\mathrm{CFT}}$.

Weakening algebras

Thus, it makes sense to require that the weakening operation itself is a pre-weakening homomorphism.

Definition

A **weakening algebra** is a pre-weakening algebra CFT with the property that $\mathbf{w}(\mathrm{CFT})$ is a pre-weakening homomorphism.

Theorem

If CFT is a weakening algebra and $p : C \to X \leftarrow Y : g$, then $Y \times_{g,p} \mathrm{CFT}$ is also a weakening algebra.

Theorem

If CFT is a weakening algebra, then so is $\mathbf{F}_{\mathrm{CFT}}$.

Rules for weakening: weakening preserves itself

The requirement that weakening is itself a pre-weakening homomorphism is represented by the following inference rules which we impose on weakening:

$$\frac{\Gamma \vdash A \ \ \mathit{fam} \quad \Gamma.B \vdash Q \ \ \mathit{fam} \quad \Gamma.B \vdash R \ \ \mathit{fam}}{((\Gamma.A).\langle A\rangle B).\langle A\rangle Q \vdash \langle A\rangle \langle Q\rangle R \equiv \langle\langle A\rangle Q\rangle \langle A\rangle R \ \ \mathit{fam}}$$

$$\frac{\Gamma \vdash A \ \ \mathit{fam} \quad \Gamma.B \vdash Q \ \ \mathit{fam} \quad (\Gamma.B).R \vdash S \ \ \mathit{fam}}{(((\Gamma.A).\langle A\rangle B).\langle A\rangle Q).\langle A\rangle \langle Q\rangle R \vdash \langle A\rangle \langle Q\rangle S \equiv \langle\langle A\rangle Q\rangle \langle A\rangle S \ \ \mathit{fam}}$$

$$\frac{\Gamma \vdash A \ \ \mathit{fam} \quad \Gamma.B \vdash Q \ \ \mathit{fam} \quad (\Gamma.B).R \vdash k : S}{(((\Gamma.A).\langle A\rangle B).\langle A\rangle Q).\langle A\rangle \langle Q\rangle R \vdash \langle A\rangle \langle Q\rangle k \equiv \langle\langle A\rangle Q\rangle \langle A\rangle k : \langle A\rangle \langle Q\rangle S}$$

Overview of the theory of projections

- In the theory of projections, we will introduce units, which will eventually behave like **fiberwise identity morphisms of families**.
- Together with weakening, units will induce all the **projections**.
- **Pre-projection algebras** will be weakening algebras with units.
- **Projection algebras** will be pre-projection algebras for which weakening is a pre-projection homomorphism.

Rules for the theory of projections

Introduction rules for units:

$$\frac{\Gamma \vdash A \;\; \mathit{fam}}{\Gamma.A \vdash \mathsf{id}_A : \langle A \rangle A} \qquad \frac{\Gamma \vdash A \equiv A' \;\; \mathit{fam}}{\Gamma.A \vdash \mathsf{id}_A \equiv \mathsf{id}_{A'} : \langle A \rangle A}$$

Weakening preserves the unit:

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma.B \vdash Q \;\; \mathit{fam}}{((\Gamma.A).\langle A \rangle B).\langle A \rangle Q \vdash \langle A \rangle \mathsf{id}_Q \equiv \mathsf{id}_{\langle A \rangle Q} : \langle \langle A \rangle Q \rangle \langle A \rangle Q}$$

Pre-projection algebras

A **pre-projection algebra** is a weakening algebra CFT for which there is a term $\mathbf{i}(\mathrm{CFT}) : F \to T_2$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\mathbf{i}(\mathrm{CFT})} & T_2 \\ {\scriptstyle \Delta_{\mathrm{ft}}}\downarrow & & \downarrow{\scriptstyle \partial_2} \\ F \times_{\mathrm{ft},\mathrm{ft}} F & \xrightarrow[w(\mathrm{CFT})_0]{} & F_2 \end{array}$$

commutes. In this diagram, $\Delta_{\mathrm{ft}} : F \to F \times_{\mathrm{ft},\mathrm{ft}} F$ is the diagonal.

Pre-projection homomorphisms

A **pre-projection homomorphism from CFT to CFT'** is a weakening homomorphism $f : \mathrm{CFT}' \to \mathrm{CFT}$ such that the square

$$\begin{array}{ccc} T_2' & \xrightarrow{f_2^t} & T_2 \\ {\scriptstyle \mathbf{i}(\mathrm{CFT}')}\uparrow & & \uparrow{\scriptstyle \mathbf{i}(\mathrm{CFT})} \\ F' & \xrightarrow[f_1]{} & F \end{array}$$

commutes.

Projection algebras

Definition

A **projection algebra** is a pre-projection algebra for which the weakening operation is a pre-projection homomorphism.

Definition

A **projection homomorphism** is a pre-projection homomorphism between projection algebras.

Theorem

If CFT *is a projection algebra and* $p : C \to X \leftarrow Y : g$*, then* $Y \times_{g,p} \mathrm{CFT}$ *is also a projection algebra.*

Theorem

If CFT *is a projection algebra, then so is* $\mathbf{F}_{\mathrm{CFT}}$*.*

Overview of the theory of substitution

- We will introduce an operation for substitution by a term, this allows us to consider **fibers of families**. The substitution of P by a will be denoted by $P[a]$.
- Substitution will be compatible with extension, thus it will be implemented as an extension homomorphism.
- **Pre-substitution algebras** will be extension algebras with a substitution operation. **Pre-substitution homomorphisms** will preserve this structure.
- **Substitution algebras** will be pre-substitution algebras for which the substitution operation itself is a pre-substitution homomorphism.

Rules for the theory of substitution

Introduction rules for substitution:

$$\frac{\Gamma \vdash a : A \quad \Gamma.A \vdash P \;\; fam}{\Gamma \vdash P[a] \;\; fam} \qquad \frac{\Gamma \vdash a \equiv a' : A \quad \Gamma.A \vdash P \equiv P' \;\; fam}{\Gamma \vdash P[a] \equiv P'[a'] \;\; fam}$$

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash Q \;\; fam}{\Gamma.(P[a]) \vdash Q[a] \;\; fam} \qquad \frac{\Gamma \vdash a \equiv a' : A \quad (\Gamma.A).P \vdash Q \equiv Q' \;\; fam}{\Gamma.(P[a]) \vdash Q[a] \equiv Q'[a'] \;\; fam}$$

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash g : Q}{\Gamma.(P[a]) \vdash g[a] : Q[a]} \qquad \frac{\Gamma \vdash a \equiv a' : A \quad (\Gamma.A).P \vdash g \equiv g' : Q}{\Gamma.(P[a]) \vdash g[a] \equiv g'[a'] : Q[a]}$$

Substitution is compatible with extension:

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash Q \;\; fam}{\Gamma \vdash (P.Q)[a] \equiv P[a].(Q[a]) \;\; fam}$$

$$\frac{\Gamma \vdash a : A \quad ((\Gamma.A).P).Q \vdash R \;\; fam}{\Gamma.(P[a]) \vdash (Q.R)[a] \equiv Q[a].(R[a]) \;\; fam}$$

Pre-substitution algebras

A **pre-substitution** for an extension algebra CFT is an extension homomorphism

$$\mathbf{s}(\mathrm{CFT}) : T \times_{\partial,\mathrm{ft}_2} \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} \to \mathbf{F}_{\mathrm{CFT}}$$

for which the square

$$\begin{array}{ccc} T \times_{\partial,\mathrm{ft}_2} \mathbf{F}_{\mathbf{F}_F} & \xrightarrow{\mathbf{s}(\mathrm{CFT})} & \mathbf{F}_{\mathrm{CFT}} \\ \Big\downarrow{\scriptstyle \partial\circ\pi_1(\partial,\mathrm{ft}_2)} & & \Big\downarrow{\scriptstyle \mathrm{ft}} \\ F & \xrightarrow[\mathrm{ft}]{} & C \end{array}$$

commutes.
A **pre-substitution algebra** is an extension algebra together with a pre-substitution.

Pre-substitution homomorphisms

A **pre-substitution homomorphism** is an extension homomorphism $f : \mathrm{CFT}' \to \mathrm{CFT}$ for which the square

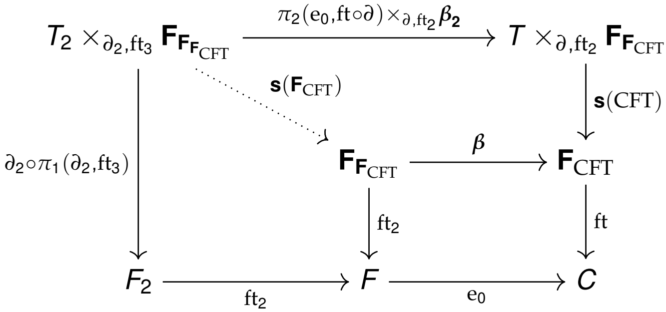
$$\begin{array}{ccc} T' \times_{\partial', \mathrm{ft}'_2} \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}'}} & \xrightarrow{f^t \times_{\partial, \mathrm{ft}_2} \mathbf{F}_{\mathbf{F}_f}} & T \times_{\partial, \mathrm{ft}_2} \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} \\ {\scriptstyle \mathbf{s}(\mathrm{CFT}')} \downarrow & & \downarrow {\scriptstyle \mathbf{s}(\mathrm{CFT})} \\ \mathbf{F}_{\mathrm{CFT}'} & \xrightarrow[\mathbf{F}_f]{} & \mathbf{F}_{\mathrm{CFT}} \end{array}$$

commutes.

The family pre-substitution algebra

Theorem

If CFT *is a pre-substitution algebra, then so is* $\mathbf{F}_{\mathrm{CFT}}$ *with* $\mathbf{s}(\mathbf{F}_{\mathrm{CFT}})$ *defined to be the unique extension homomorphism rendering the diagram*

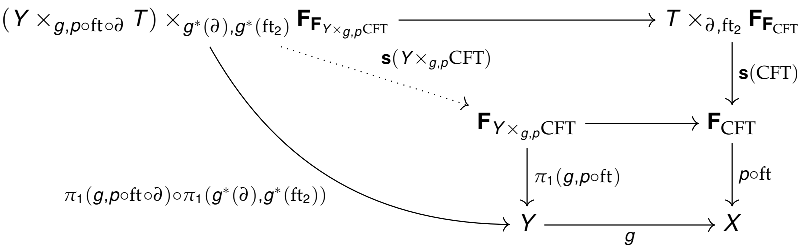


commutative.

Change of base of pre-substitution algebras

Theorem

Let CFT *be a pre-substitution algebra and consider* $p : C \to X \leftarrow Y : g$*. Then* $Y \times_{g,p} \mathrm{CFT}$ *is a pre-substitution algebra with* $\mathbf{s}(Y \times_{g,p} \mathrm{CFT})$ *defined to be the unique extension homomorphism rendering the diagram*



commutative.

Substitution algebras

Thus, it makes sense to require that the pre-substitution operation itself is a pre-substitution homomorphism.

Definition

A **substitution algebra** is a pre-substitution algebra CFT with the property that $\mathbf{S}(\mathrm{CFT})$ is a pre-substitution morphism.

Definition

A **substitution homomorphism** is a pre-substitution homomorphism between substitution algebras.

Theorem

If CFT *is a substitution algebra and* $p : C \to X \leftarrow Y : g$*, then* $Y \times_{g,p} \mathrm{CFT}$ *is also a substitution algebra.*

Theorem

If CFT *is a substitution algebra, then so is* $\mathbf{F}_{\mathrm{CFT}}$*.*

Rules for substitution: substitution is compatible with itself

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash g : Q \quad ((\Gamma.A).P).Q \vdash R \;\; \mathit{fam}}{(\Gamma.(P[a])).(Q[a]) \vdash R[g][a] \equiv R[a][g[a]] \;\; \mathit{fam}}$$

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash g : Q \quad (((\Gamma.A).P).Q).R \vdash S \;\; \mathit{fam}}{((\Gamma.(P[a])).(Q[a])).(R[g][a]) \vdash S[g][a] \equiv S[a][g[a]] \;\; \mathit{fam}}$$

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash g : Q \quad (((\Gamma.A).P).Q).R \vdash k : S}{((\Gamma.(P[a])).(Q[a])).(R[g][a]) \vdash k[g][a] \equiv k[a][g[a]] : S[g][a]}$$

Joining the theories of projections and substitution

To join the two theories we have formulated on top of the theory of extension, we need to provide rules describing:

- that weakening preserves substitution (weakening is a substitution homomorphism);
- that substitution preserves weakening (substitution is a weakening homomorphism);
- that substitution preserves units (substitution is a projection homomorphism);
- that weakenings are constant families;
- everything is invariant with respect to precomposition with units.

Pre-E-systems

A **pre-E-system** is an extension algebra CFT with a weakening operation $\mathbf{w}(\mathrm{CFT})$, units $\mathbf{i}(\mathrm{CFT})$ and a substitution operation $\mathbf{s}(\mathrm{CFT})$ giving it both the structure of a projection algebra and a substitution algebra, such that in addition

- Weakening is a substitution homomorphism;
- Substitution is a projection homomorphism.

Pre-E-homomorphisms are extension homomorphisms which are both projection homomorphisms and substitution homomorphisms.

Rules: Weakened families are constant families

$$\frac{\Gamma \vdash A \ \ fam \quad \Gamma \vdash B \ \ fam \quad \Gamma \vdash a : A}{\Gamma \vdash (\langle A \rangle B)[a] \equiv B \ \ fam}$$

$$\frac{\Gamma \vdash A \ \ fam \quad \Gamma.B \vdash Q \ \ fam \quad \Gamma \vdash a : A}{\Gamma.B \vdash (\langle A \rangle Q)[a] \equiv Q \ \ fam}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma.B \vdash g : Q}{\Gamma.B \vdash (\langle A \rangle g)[a] \equiv g : Q}$$

Weakened families are constant families

In a pre-E-system, we say that **weakened families are constant families** if the diagram

$$\begin{array}{ccc} T \times_{\mathrm{ft}\circ\partial,\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}} & \xrightarrow{[\mathrm{id}_T, \mathbf{w}(\mathrm{CFT})\circ(\partial \times_{\mathrm{ft},\mathrm{ft}} \mathrm{id}_{\mathbf{F}_{\mathrm{CFT}}})]} & T \times_{\partial,\mathrm{ft}_2} \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} \\ & \searrow^{\pi_2(\mathrm{ft}\circ\partial,\mathrm{ft})} & \downarrow \mathbf{s}(\mathrm{CFT}) \\ & & \mathbf{F}_{\mathrm{CFT}} \end{array}$$

commutes.

Rules: precomposition with a unit has no effect

$$\frac{\Gamma.A \vdash P \;\; \mathit{fam}}{\Gamma.A \vdash (\langle A \rangle P)[\mathsf{id}_A] \equiv P \;\; \mathit{fam}}$$

$$\frac{(\Gamma.A).P \vdash Q \;\; \mathit{fam}}{(\Gamma.A).P \vdash (\langle A \rangle Q)[\mathsf{id}_A] \equiv Q \;\; \mathit{fam}}$$

$$\frac{(\Gamma.A).P \vdash g : Q}{(\Gamma.A).P \vdash (\langle A \rangle g)[\mathsf{id}_A] \equiv g : Q}$$

Invariance with respect to precomposition with units

In a pre-E-system CFT we say that **everything is invariant with respect to precomposition with units** if the diagram

$$
\begin{array}{ccccc}
\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} & \longrightarrow & \mathbf{F}_{F \times_{\mathrm{ft},\mathrm{ft}} \mathbf{F}_{\mathrm{CFT}}} & \xrightarrow{[\pi_1(\mathrm{ft},\mathrm{ft}) \circ \mathrm{ft}^*(\mathrm{ft}_2), \mathbf{F}_{\mathbf{w}(\mathrm{CFT})}]} & F \times_{w(\mathrm{CFT})_0 \circ \Delta_{\mathrm{ft}}, \mathrm{ft}_3} \mathbf{F}_{\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}} \\
 & & & & \Big\downarrow{\scriptstyle \mathbf{i}(\mathrm{CFT}) \times_{\partial_2,\mathrm{ft}_3} \mathrm{id}_{\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}}]} \\
 & & & & T_2 \times_{\partial_2,\mathrm{ft}_3} \mathbf{F}_{\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}} \\
 & & & & \Big\downarrow{\scriptstyle \mathbf{s}(\mathbf{F}_{\mathrm{CFT}})} \\
\mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}} & & = & & \mathbf{F}_{\mathbf{F}_{\mathrm{CFT}}}
\end{array}
$$

commutes.

E-systems

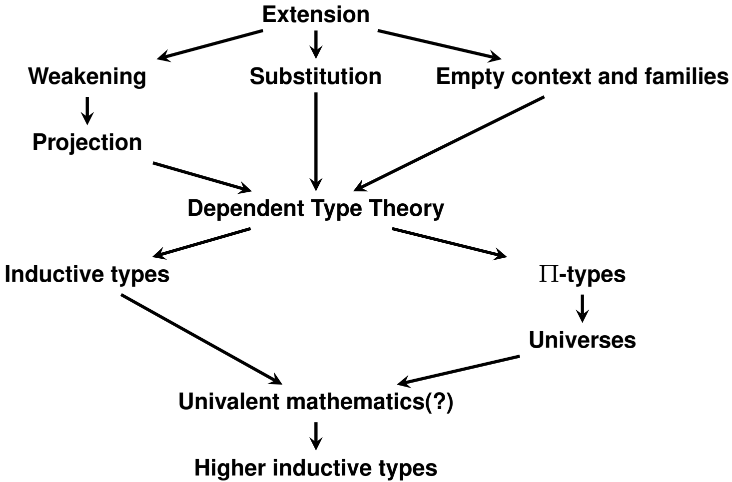
An **E-system** is an extension algebra with

- a weakening operation and units, giving it the structure of a projection algebra;
- a substitution operation, giving it the structure of a substitution algebra;
- (an empty context and empty families);

such that

- Weakening is a substitution homomorphism;
- Substitution is a projection homomorphism;
- Weakened families are constant families;
- Precomposition with identity functions leaves everything invariant.
- (The requirements regarding the empty context and families according to their rules)

Towards algebraic Homotopy Type Theory



Appendix: weakening and substitution preserve each other

Weakening preserves substitution:

$$\frac{\Gamma \vdash A \ \ \mathit{fam} \quad \Gamma.B \vdash g : Q \quad (\Gamma.B).Q \vdash R \ \ \mathit{fam}}{(\Gamma.A).\langle A\rangle B \vdash \langle A\rangle(R[g]) \equiv (\langle A\rangle R)[\langle A\rangle g] \ \ \mathit{fam}}$$

$$\frac{\Gamma \vdash A \ \ \mathit{fam} \quad \Gamma.B \vdash g : Q \quad ((\Gamma.B).Q).R \vdash S \ \ \mathit{fam}}{((\Gamma.A).\langle A\rangle B).\langle A\rangle(R[g]) \vdash \langle A\rangle(S[g]) \equiv (\langle A\rangle S)[\langle A\rangle g] \ \ \mathit{fam}}$$

$$\frac{\Gamma \vdash A \ \ \mathit{fam} \quad \Gamma.B \vdash g : Q \quad ((\Gamma.B).Q).R \vdash k : S}{((\Gamma.A).\langle A\rangle B).\langle A\rangle(R[g]) \vdash \langle A\rangle(k[g]) \equiv (\langle A\rangle k)[\langle A\rangle g] : \langle A\rangle(S[g])}$$

Substitution preserves weakening:

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash Q \ \ \mathit{fam} \quad (\Gamma.A).P \vdash R \ \ \mathit{fam}}{(\Gamma.(P[a])).(Q[a]) \vdash (\langle Q\rangle R)[a] \equiv \langle Q[a]\rangle(R[a]) \ \ \mathit{fam}}$$

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash Q \ \ \mathit{fam} \quad ((\Gamma.A).P).R \vdash S \ \ \mathit{fam}}{((\Gamma.(P[a])).(Q[a])).((\langle Q\rangle R)[a]) \vdash (\langle Q\rangle S)[a] \equiv \langle Q[a]\rangle(S[a]) \ \ \mathit{fam}}$$

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash Q \ \ \mathit{fam} \quad ((\Gamma.A).P).R \vdash k : S}{((\Gamma.(P[a])).(Q[a])).((\langle Q\rangle R)[a]) \vdash (\langle Q\rangle k)[a] \equiv \langle Q[a]\rangle(k[a]) : (\langle Q\rangle S)[a]}$$

Appendix: substitution preserves units

$$\frac{\Gamma \vdash a : A \quad (\Gamma.A).P \vdash Q \;\; \mathit{fam}}{(\Gamma.(P[a])).(Q[a]) \vdash \mathsf{id}_Q[a] \equiv \mathsf{id}_{Q[a]} : \langle Q[a] \rangle Q[a]}$$

Appendix: the empty context and family

Introduction rules for the empty context and family:

$$\frac{}{\vdash [\,]\ \ \mathit{ctx}} \qquad \frac{\vdash \Gamma\ \ \mathit{ctx}}{\Gamma \vdash [\,]_\Gamma\ \ \mathit{fam}} \qquad \frac{\vdash \Gamma \equiv \Gamma'\ \ \mathit{ctx}}{\Gamma \vdash [\,]_\Gamma \equiv [\,]_{\Gamma'}\ \ \mathit{fam}}$$

Every context induces a family in the empty context:

$$\frac{\vdash \Gamma\ \ \mathit{ctx}}{[\,] \vdash i(\Gamma)\ \ \mathit{fam}} \qquad \frac{\vdash \Gamma \equiv \Delta\ \ \mathit{ctx}}{[\,] \vdash i(\Gamma) \equiv i(\Delta)\ \ \mathit{fam}}$$

Compatibility rules for empty context and family with extension:

$$\frac{\vdash \Gamma\ \ \mathit{ctx}}{\vdash [\,].i(\Gamma) \equiv \Gamma\ \ \mathit{ctx}} \qquad \frac{\vdash \Gamma\ \ \mathit{ctx}}{\vdash \Gamma.[\,] \equiv \Gamma\ \ \mathit{ctx}}$$

$$\frac{\Gamma \vdash A\ \ \mathit{fam}}{\Gamma \vdash [\,].A \equiv A\ \ \mathit{fam}} \qquad \frac{\Gamma \vdash A\ \ \mathit{fam}}{\Gamma \vdash A.[\,] \equiv A\ \ \mathit{fam}}$$

$$\frac{[\,] \vdash A\ \ \mathit{fam}}{[\,] \vdash i([\,].A) \equiv A\ \ \mathit{fam}}$$

and finally

$$\frac{\Gamma \vdash A\ \ \mathit{fam}}{\vdash \Gamma.A \equiv i(\Gamma).A\ \ \mathit{ctx}} \qquad \frac{}{[\,] \vdash i([\,]) \equiv [\,]_{[\,]}\ \ \mathit{fam}}$$

Appendix: weakening and the empty families

Weakening by the empty family:

$$\frac{\Gamma \vdash B \;\; \mathit{fam}}{\Gamma \vdash \langle [\,] \rangle B \equiv B \;\; \mathit{fam}}$$

$$\frac{\Gamma.B \vdash Q \;\; \mathit{fam}}{\Gamma.B \vdash \langle [\,] \rangle Q \equiv Q \;\; \mathit{fam}}$$

$$\frac{\Gamma.B \vdash g : Q}{\Gamma.B \vdash \langle [\,] \rangle g \equiv g : Q}$$

Weakening of the empty family:

$$\frac{\Gamma \vdash A \;\; \mathit{fam}}{\Gamma.A \vdash \langle A \rangle [\,] \equiv [\,] \;\; \mathit{fam}}$$

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma \vdash B \;\; \mathit{fam}}{(\Gamma.A).\langle A \rangle B \vdash \langle A \rangle [\,] \equiv [\,] \;\; \mathit{fam}}$$

Compatibility of weakening of a family with weakening of a context:

$$\frac{\Gamma \vdash A \;\; \mathit{fam} \quad \Gamma \vdash B \;\; \mathit{fam}}{\Gamma.A \vdash \langle A \rangle^{\mathit{ctx}} B \equiv \langle A \rangle^{\mathit{fam}} B \;\; \mathit{fam}}$$

Appendix: substitution of an empty family

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash [\,][a] \equiv [\,] \;\; \mathit{fam}}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma.A \vdash P \;\; \mathit{fam}}{\Gamma.(P[a]) \vdash [\,][a] \equiv [\,] \;\; \mathit{fam}}$$

Compatibility of substitution of a family with substitution of a context:

$$\frac{\Gamma.A \vdash P \;\; \mathit{fam}}{\Gamma \vdash P[a]^{\mathit{ctx}} \equiv P[a]^{\mathit{fam}} \;\; \mathit{fam}}$$

Appendix: units act as identity functions

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{id}_A[a] \equiv a : A}$$

Note that the empty family is needed to have this rule.