Subsumption of concepts in $\mathcal{FL}_0$ for (cyclic) terminologies with respect to descriptive semantics is PSPACE-complete.

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joint work with Hans de Nivelle

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- $\mathcal{A}$ – atomic concepts *(unary relations)*
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The reasoning tasks in DL

Knowledge base (or terminology) $\mathcal{T}$:

$$Human \doteq Mammal \sqcap \forall parent.Human$$

$$Elephant \doteq Mammal \sqcap \forall parent.Elephant$$

$$Adam \doteq Mammal \sqcap \forall parent.\bot$$

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Should really all models satisfying $\mathcal{T}$ be considered?

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- gfp-semantics: cyclic definitions are evaluated in maximal possible way (*“all”-definitions*):

$$MOMO \doteq Man \sqcap \forall child.MOMO$$

The small terminological language

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Subsumption in $\mathcal{FL}_0$	Cyclic T-Boxes	Acyclic T-Boxes
descriptive semantics	in PSPACE, **PSPACE-hard**	co-NP-complete
lfp-semantics	PSPACE-complete	
gfp-semantics	PSPACE-complete	

The description graph

We focus our attention on terminologies $\mathcal{T}$ of the form:

$$A_i \doteq \forall R_{i,1}.B_{i,1} \sqcap \cdots \sqcap \forall R_{i,k_i}.B_{i,k_i} \tag{1}$$

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The description graph $\mathcal{G}_{\mathcal{T}}$ is a graph, where:

- Nodes are labelled by concept names;
- Oriented edges are labelled by role names such that: the edge e comes from the node n_1 to the node n_2 iff
  - n_1 is labelled by A, n_2 is labelled by B,
  - e is labelled by R and
  - $A \doteq \cdots \sqcap \forall R.B \sqcap \cdots \in \mathcal{T}$.

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Consider the terminology $\mathcal{T}$:

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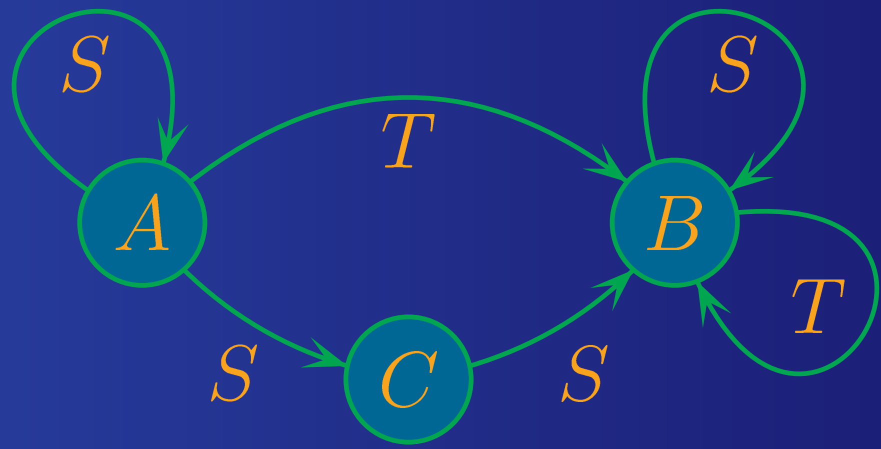

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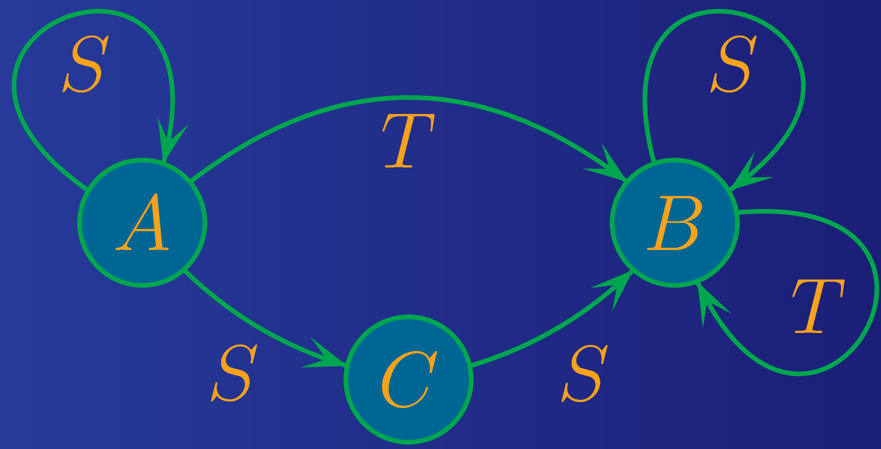

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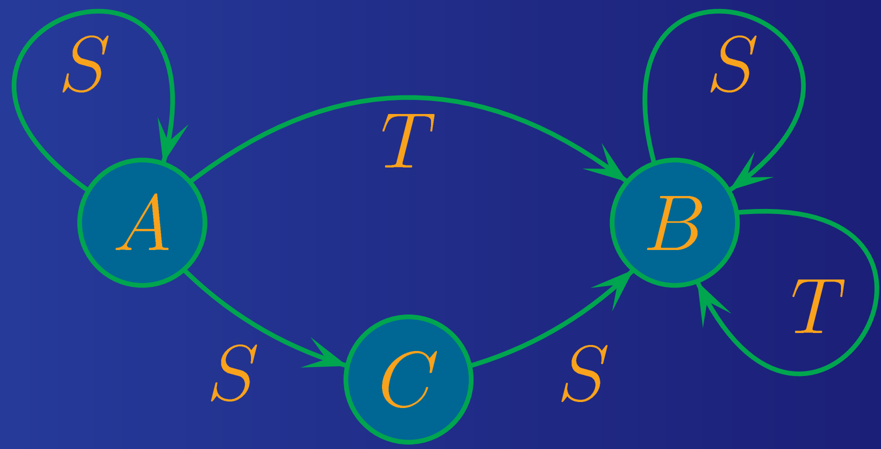

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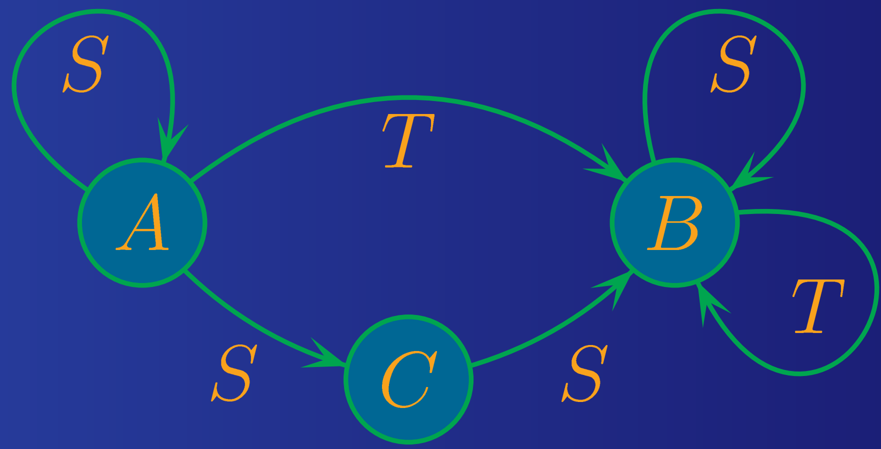

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- Repeating, we construct the infinite sequence of $a_i \in (\neg B)^{\mathcal{M}}$ with $(a_i, a_{i+1}) \in S^{\mathcal{M}}$ or $\in T^{\mathcal{M}}$.

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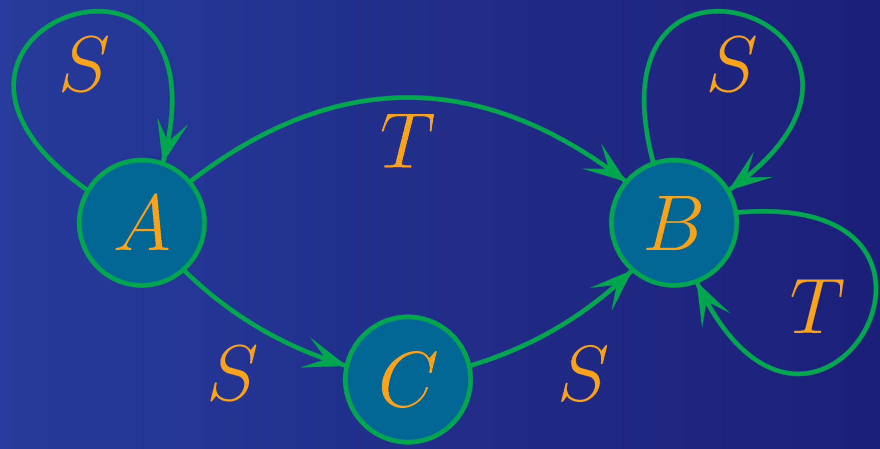

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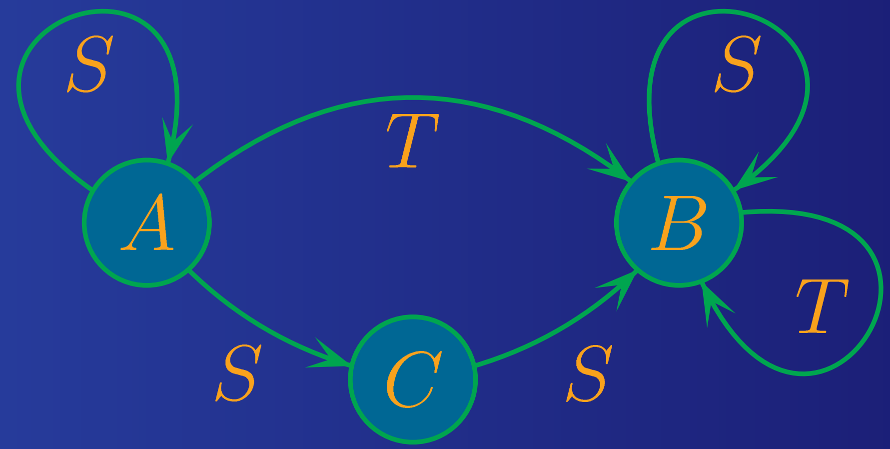

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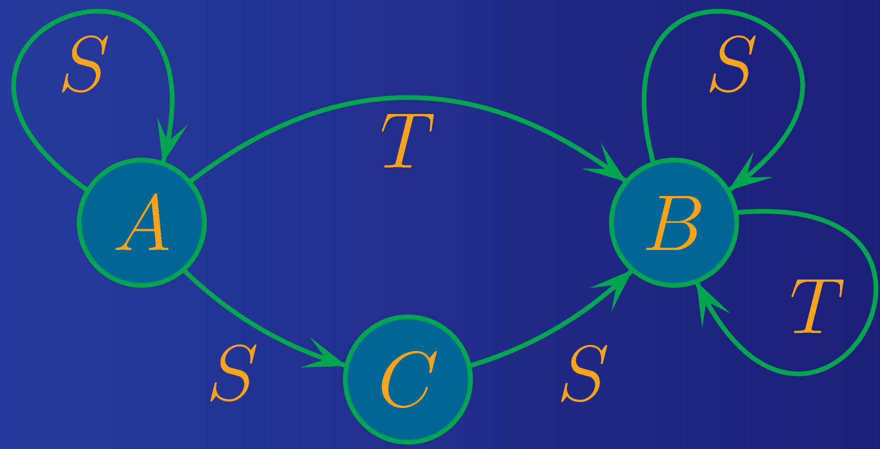

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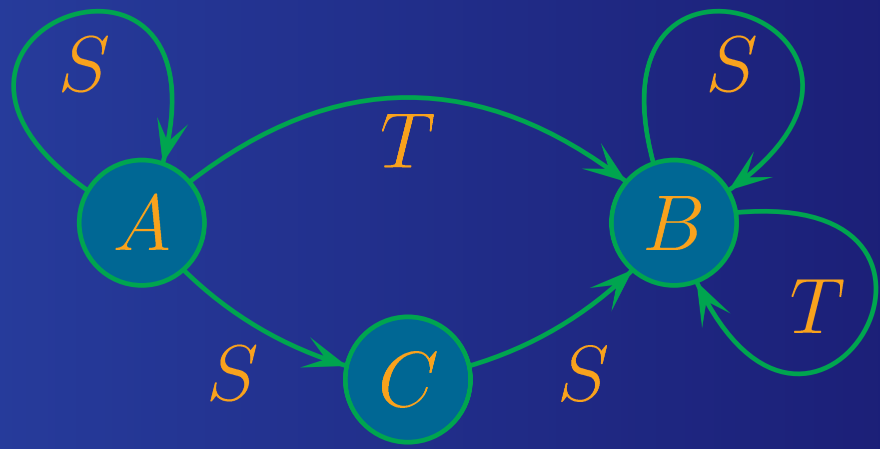

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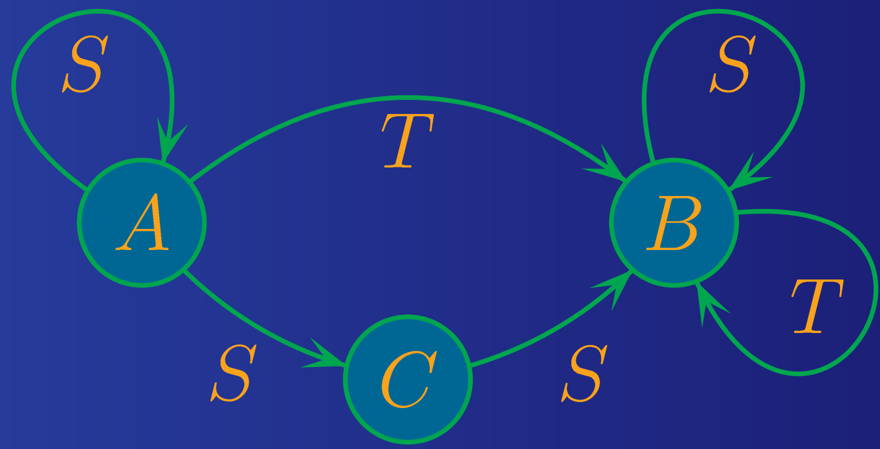

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- Since $a_i \in (\neg B)^{\mathcal{M}}$, no such $\mathcal{M}$ exists, thus $A \sqsubseteq_{\mathcal{T}} B$.

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The following can be shown using the similar arguments:

Lemma. (Characterization of concept subsumption)

$A \sqsubseteq_{\mathcal{T}} B$ iff in the description graph $\mathcal{G}_{\mathcal{T}}$
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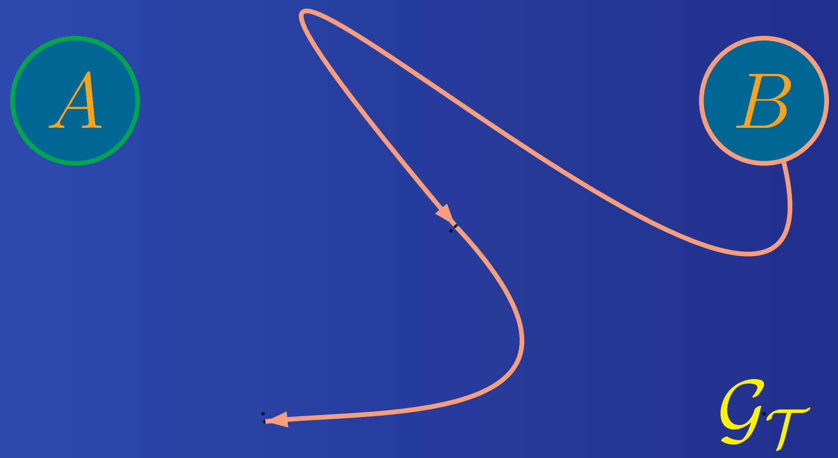

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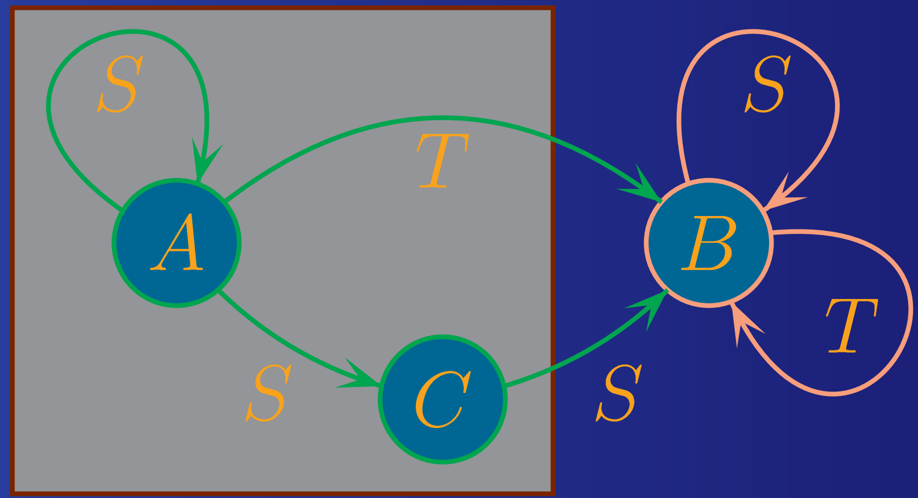

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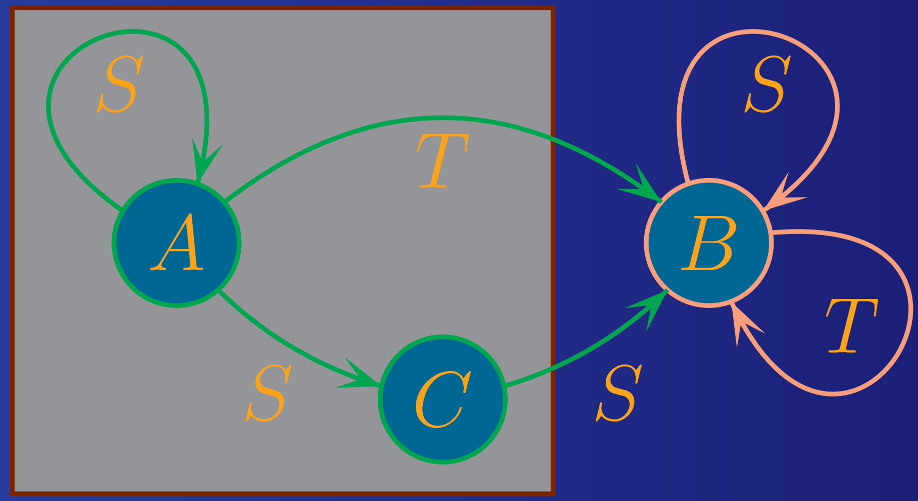

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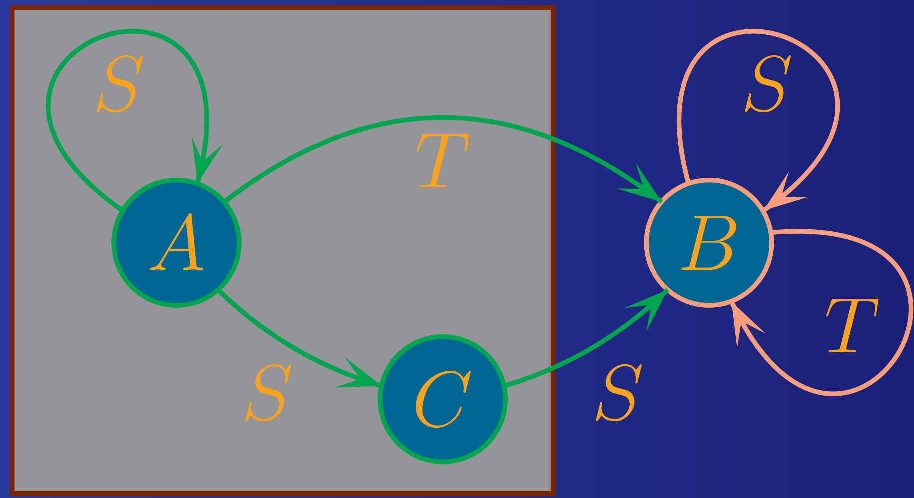

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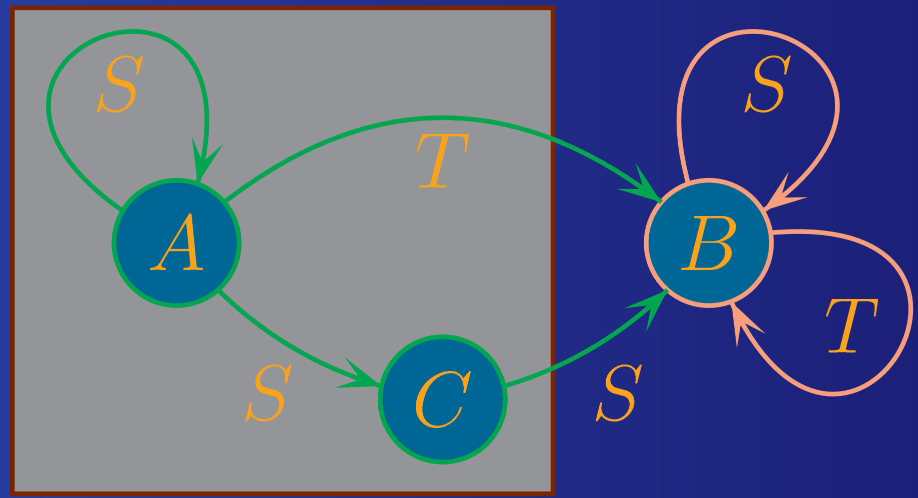

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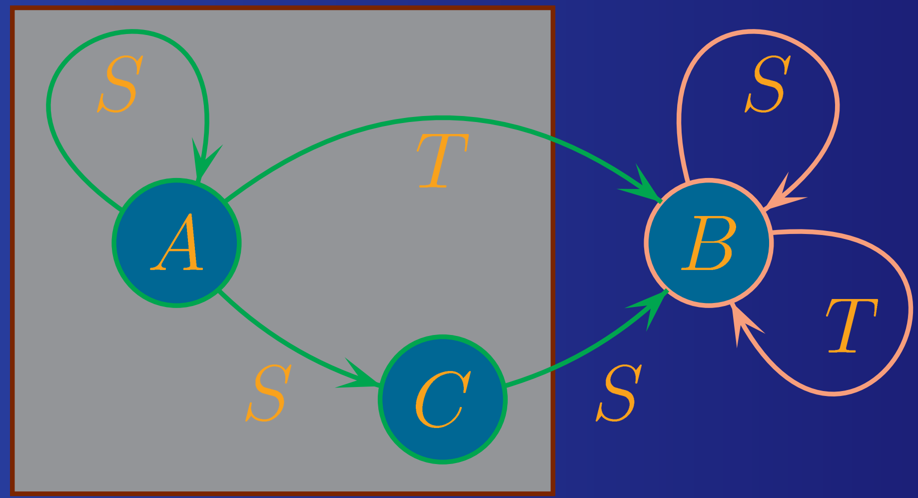

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- Thus $A \sqsubseteq_{\mathcal{T}} B$ and $C \not\sqsubseteq_{\mathcal{T}} B$

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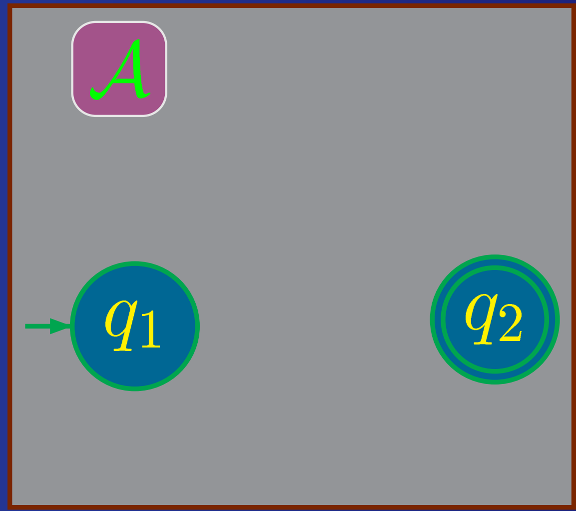

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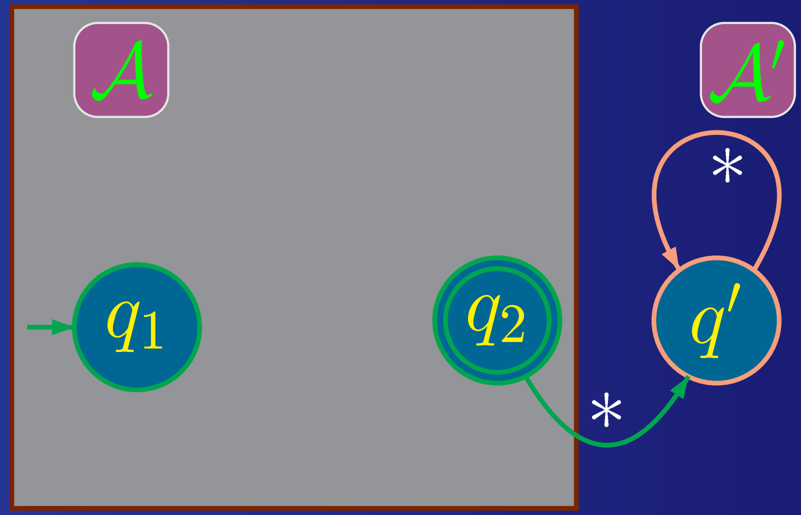

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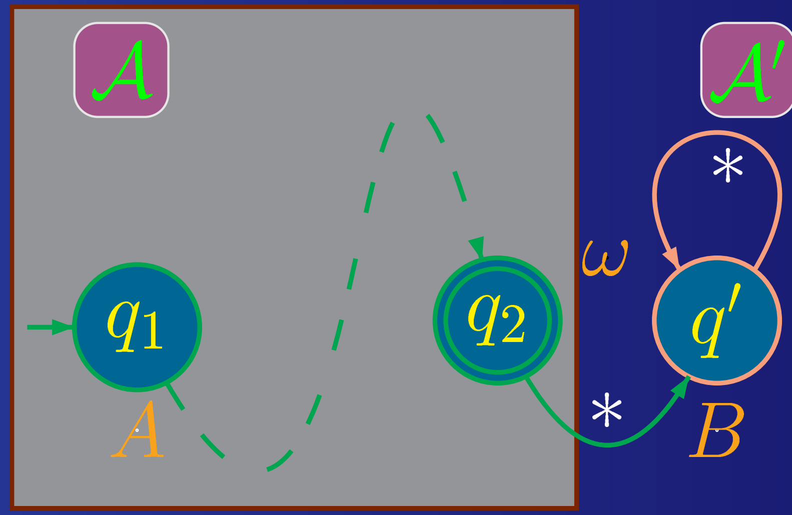

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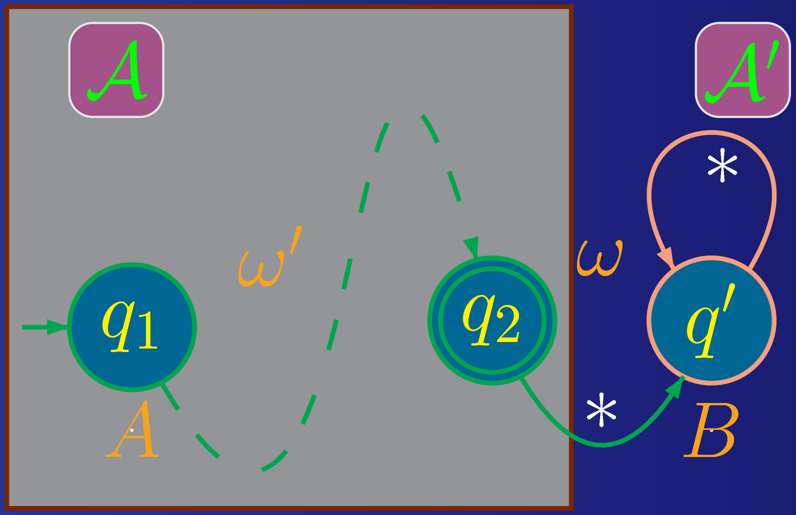

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- $A \sqsubseteq_{\mathcal{T}} B$ iff for any word $\omega \in \Sigma^\omega$ there is a finite prefix ω' which is accepted by $\mathcal{A}$.

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- It is reasonable to view our problem as the universality problem for automata with prefix acceptance condition.

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- We introduce a prefix acceptance condition for $\mathbf{NFA}$:
  "An infinite word is accepted iff $\mathbf{NFA}$ *accepts it's finite prefix."*
- The problem: *"Given* $\mathbf{NFA}_p^\omega$ *check whether it accepts every infinite word"*. How hard is it?
- Similar problem for finite automata and bučhi automata is known as the universality problem.
- It is PSPACE-complete
- It is reasonable to view our problem as the universality problem for automata with prefix acceptance condition.
- The alternative formulation for the problem:

$$\boxed{L \cdot \Sigma^\omega = \Sigma^\omega \ ?}$$

The main result

Lemma.(The reduction lemma)

Concept subsumption for descriptive semantics is not easy than the universality problem for automata with prefix acceptance condition.

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The universality problem for $\mathbf{NFA}_p^\omega$ *is PSPACE-complete.*

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Theorem.

The universality problem for $\mathbf{NFA}_p^{\omega}$ is PSPACE-complete.

Corollary

Subsumption of concepts in $\mathcal{FL}_0$ for (cyclic) terminologies with respect to descriptive semantics is PSPACE-complete.

Conclusions and related work

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Conclusions and related work

- We have confirmed the relationship between subsumption problem and automata-theoretic problems.
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$$\mathcal{EL} ::= \ A \mid C_1 \sqcap C_2 \mid \exists R.C.$$
Franz Baader (2002):
Subsumption in $\mathcal{EL}$ is polynomial.
- Description logics with mixed semantics?
T.Henzinger, O. Kupferman, R.Majumdar (2003):
satisfiability of $\forall MC$ is PSPACE-complete,
satisfiability of $\exists MC$ is NP-complete.
However, implication problem ($\sim$ subsumption) is still EXPTIME.

Thank you!

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