Appendix A

Some laws for predicate calculation

This collection of laws is drawn from [MS89].

A.1 Some propositional laws

Throughout this section $\mathcal{A}$, $\mathcal{B}$ are $\mathcal{C}$ denote formulae of predicate calculus. The laws are *propositional* because they do not deal with the quantification or substitution of variables.

A.1.1 Conjunction and disjunction

The propositional connectives for conjunction, $\wedge$, and disjunction, $\vee$, are idempotent, commutative, associative and absorptive, and they distribute through each other.

Idempotence of $\wedge$ *and* $\vee$
Conjunction and disjunction are *idempotent* connectives:

$$\mathcal{A} \wedge \mathcal{A} \quad \equiv \quad \mathcal{A} \quad \equiv \quad \mathcal{A} \vee \mathcal{A} \,. \tag{A.1}$$

Commutativity of $\wedge$ *and* $\vee$
Conjunction and disjunction are *commutative* connectives:

$$\mathcal{A} \wedge \mathcal{B} \;\equiv\; \mathcal{B} \wedge \mathcal{A} \tag{A.2}$$

$$\mathcal{A} \vee \mathcal{B} \;\equiv\; \mathcal{B} \vee \mathcal{A} \,. \tag{A.3}$$

Associativity of $\wedge$ *and* $\vee$
Conjunction and disjunction are *associative* connectives:

$$\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) \;\equiv\; (\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C} \tag{A.4}$$

$$\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C}) \quad \equiv \quad (\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C} \ . \tag{A.5}$$

Laws A.1 to A.5 mean that we can ignore duplication, order and bracketing in conjunctions $\mathcal{A} \wedge \mathcal{B} \wedge \cdots \wedge \mathcal{C}$ and disjunctions $\mathcal{A} \vee \mathcal{B} \vee \cdots \vee \mathcal{C}$.

Absorption laws
Sometimes terms can be removed immediately from expressions involving both conjunctions and disjunctions. This is *absorption*:

$$\mathcal{A} \wedge (\mathcal{A} \vee \mathcal{B}) \quad \equiv \quad \mathcal{A} \quad \equiv \quad \mathcal{A} \vee (\mathcal{A} \wedge \mathcal{B}) \ . \tag{A.6}$$

Distributive laws
The *distribution* of $\wedge$ through $\vee$ is similar to the distribution of multiplication over addition in arithmetic. But in logic distribution goes both ways, so that $\vee$ also distributes through $\wedge$:

$$\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) \quad \equiv \quad (\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C}) \tag{A.7}$$

$$\mathcal{A} \vee (\mathcal{B} \wedge \mathcal{C}) \quad \equiv \quad (\mathcal{A} \vee \mathcal{B}) \wedge (\mathcal{A} \vee \mathcal{C}) \ . \tag{A.8}$$

A.1.2 Constants and negation

Units and zeroes
In ordinary multiplication, $a \times 1 = a$ and $a \times 0 = 0$. We say therefore that 1 is a *unit* and 0 a *zero* of multiplication. Similarly, the predicate constant true is the unit of $\wedge$ and the zero of $\vee$:

$$\mathcal{A} \wedge \mathsf{true} \quad \equiv \quad \mathcal{A} \tag{A.9}$$

$$\mathcal{A} \vee \mathsf{true} \quad \equiv \quad \mathsf{true} \ . \tag{A.10}$$

The constant false is the unit of $\vee$ and the zero of $\wedge$:

$$\mathcal{A} \wedge \mathsf{false} \quad \equiv \quad \mathsf{false} \tag{A.11}$$

$$\mathcal{A} \vee \mathsf{false} \quad \equiv \quad \mathcal{A} \ . \tag{A.12}$$

Negation as complement
Negation $\neg$ acts as a *complement*:

$$\neg\mathsf{true} \quad \equiv \quad \mathsf{false} \tag{A.13}$$

$$\neg\mathsf{false} \quad \equiv \quad \mathsf{true} \tag{A.14}$$

$$\mathcal{A} \wedge \neg\mathcal{A} \quad \equiv \quad \mathsf{false} \tag{A.15}$$

$$\mathcal{A} \vee \neg\mathcal{A} \quad \equiv \quad \mathsf{true} \ . \tag{A.16}$$

Furthermore it is an *involution*:

$$\neg\neg\mathcal{A} \quad \equiv \quad \mathcal{A} \ . \tag{A.17}$$

And it satisfies de Morgan's laws:

$$\neg(\mathcal{A} \wedge \mathcal{B}) \equiv \neg\mathcal{A} \vee \neg\mathcal{B} \tag{A.18}$$
$$\neg(\mathcal{A} \vee \mathcal{B}) \equiv \neg\mathcal{A} \wedge \neg\mathcal{B} \;. \tag{A.19}$$

Further absorptive laws
With negation, we have two more absorptive laws:

$$\mathcal{A} \vee (\neg\mathcal{A} \wedge \mathcal{B}) \equiv \mathcal{A} \vee \mathcal{B} \tag{A.20}$$
$$\mathcal{A} \wedge (\neg\mathcal{A} \vee \mathcal{B}) \equiv \mathcal{A} \wedge \mathcal{B} \;. \tag{A.21}$$

A.1.3 Normal forms

A formula is in *disjunctive normal form* if it is a finite disjunction of other formulae each of which is, in turn, a *conjunction* of simple formulae. *Conjunctive normal form* is defined complementarily.

Laws A.7, A.8, A.18 and A.19 allow us to convert any proposition to either disjunctive or conjunctive normal form, as we choose, and laws A.15 and A.16 serve to remove adjacent complementary formulae. For example,

$$\begin{array}{ll}
 & \mathcal{A} \wedge \neg(\mathcal{B} \wedge \mathcal{C} \wedge \mathcal{A}) \\
\equiv & \text{“Predicate law A.18”} \\
 & \mathcal{A} \wedge (\neg\mathcal{B} \vee \neg\mathcal{C} \vee \neg\mathcal{A}) \\
\equiv & \text{“Predicate law A.7”} \\
 & (\mathcal{A} \wedge \neg\mathcal{B}) \vee (\mathcal{A} \wedge \neg\mathcal{C}) \vee (\mathcal{A} \wedge \neg\mathcal{A}) \\
\equiv & \text{“Predicate law A.15”} \\
 & (\mathcal{A} \wedge \neg\mathcal{B}) \vee (\mathcal{A} \wedge \neg\mathcal{C}) \vee \mathsf{false} \\
\equiv & \text{“Predicate law A.12”} \\
 & (\mathcal{A} \wedge \neg\mathcal{B}) \vee (\mathcal{A} \wedge \neg\mathcal{C}) \;.
\end{array}$$

The second formula above is in conjunctive normal form and the third, fourth, and fifth are in disjunctive normal form.

A.1.4 Implication

Implication $\Rightarrow$ satisfies the law

$$\mathcal{A} \Rightarrow \mathcal{B} \equiv \neg\mathcal{A} \vee \mathcal{B} \;, \tag{A.22}$$

and that leads on to these laws:

$$\mathcal{A} \Rightarrow \mathcal{A} \equiv \mathsf{true} \tag{A.23}$$
$$\mathcal{A} \Rightarrow \mathcal{B} \equiv \neg(\mathcal{A} \wedge \neg\mathcal{B}) \tag{A.24}$$
$$\neg(\mathcal{A} \Rightarrow \mathcal{B}) \equiv \mathcal{A} \wedge \neg\mathcal{B} \tag{A.25}$$
$$\mathcal{A} \Rightarrow \mathcal{B} \equiv \neg\mathcal{B} \Rightarrow \neg\mathcal{A} \;. \tag{A.26}$$

The last above is called the *contrapositive law*. Useful special cases of those are

$$\begin{aligned} \mathcal{A} \Rightarrow \mathsf{true} &\equiv \mathsf{true} && \text{(A.27)}\\ \mathsf{true} \Rightarrow \mathcal{A} &\equiv \mathcal{A} && \text{(A.28)}\\ \mathcal{A} \Rightarrow \mathsf{false} &\equiv \neg\mathcal{A} && \text{(A.29)}\\ \mathsf{false} \Rightarrow \mathcal{A} &\equiv \mathsf{true} && \text{(A.30)}\\ \mathcal{A} \Rightarrow \neg\mathcal{A} &\equiv \neg\mathcal{A} && \text{(A.31)}\\ \neg\mathcal{A} \Rightarrow \mathcal{A} &\equiv \mathcal{A}\ . && \text{(A.32)} \end{aligned}$$

These next two laws distribute implication $\Rightarrow$ through conjunction and disjunction:

$$\begin{aligned} \mathcal{C} \Rightarrow (\mathcal{A} \wedge \mathcal{B}) &\equiv (\mathcal{C} \Rightarrow \mathcal{A}) \wedge (\mathcal{C} \Rightarrow \mathcal{B}) && \text{(A.33)}\\ (\mathcal{A} \vee \mathcal{B}) \Rightarrow \mathcal{C} &\equiv (\mathcal{A} \Rightarrow \mathcal{C}) \wedge (\mathcal{B} \Rightarrow \mathcal{C}) && \text{(A.34)}\\ \mathcal{C} \Rightarrow (\mathcal{A} \vee \mathcal{B}) &\equiv (\mathcal{C} \Rightarrow \mathcal{A}) \vee (\mathcal{C} \Rightarrow \mathcal{B}) && \text{(A.35)}\\ (\mathcal{A} \wedge \mathcal{B}) \Rightarrow \mathcal{C} &\equiv (\mathcal{A} \Rightarrow \mathcal{C}) \vee (\mathcal{B} \Rightarrow \mathcal{C})\ . && \text{(A.36)} \end{aligned}$$

Extra laws of implication
The following laws are useful in showing that successive hypotheses may be conjoined or even reversed:

$$\mathcal{A} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C}) \quad\equiv\quad (\mathcal{A} \wedge \mathcal{B}) \Rightarrow \mathcal{C} \quad\equiv\quad \mathcal{B} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{C})\ . \tag{A.37}$$

And the next law is the basis of definition by cases:

$$(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\neg\mathcal{A} \Rightarrow \mathcal{C}) \quad\equiv\quad (\mathcal{A} \wedge \mathcal{B}) \vee (\neg\mathcal{A} \wedge \mathcal{C})\ . \tag{A.38}$$

A.1.5 Equivalence

Equivalence satisfies this law:

$$\begin{aligned} \mathcal{A} \Leftrightarrow \mathcal{B} &\equiv (\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{B} \Rightarrow \mathcal{A}) && \text{(A.39)}\\ &\equiv (\mathcal{A} \wedge \mathcal{B}) \vee \neg(\mathcal{A} \vee \mathcal{B}) && \text{(A.40)}\\ &\equiv \neg\mathcal{A} \Leftrightarrow \neg\mathcal{B}\ . && \text{(A.41)} \end{aligned}$$

Also we have these:

$$\begin{aligned} \mathcal{A} \Leftrightarrow \mathcal{A} &\equiv \mathsf{true} && \text{(A.42)}\\ \mathcal{A} \Leftrightarrow \neg\mathcal{A} &\equiv \mathsf{false} && \text{(A.43)}\\ \mathcal{A} \Leftrightarrow \mathsf{true} &\equiv \mathcal{A} && \text{(A.44)}\\ \mathcal{A} \Leftrightarrow \mathsf{false} &\equiv \neg\mathcal{A} && \text{(A.45)}\\ \mathcal{A} \Rightarrow \mathcal{B} &\equiv \mathcal{A} \Leftrightarrow (\mathcal{A} \wedge \mathcal{B}) && \text{(A.46)}\\ \mathcal{B} \Rightarrow \mathcal{A} &\equiv \mathcal{A} \Leftrightarrow (\mathcal{A} \vee \mathcal{B}) && \text{(A.47)}\\ \mathcal{A} \vee (\mathcal{B} \Leftrightarrow \mathcal{C}) &\equiv (\mathcal{A} \vee \mathcal{B}) \Leftrightarrow (\mathcal{A} \vee \mathcal{C})\ . && \text{(A.48)} \end{aligned}$$

Equivalence is commutative and associative

$$\mathcal{A} \Leftrightarrow \mathcal{B} \;\equiv\; \mathcal{B} \Leftrightarrow \mathcal{A} \tag{A.49}$$

$$\mathcal{A} \Leftrightarrow (\mathcal{B} \Leftrightarrow \mathcal{C}) \;\equiv\; (\mathcal{A} \Leftrightarrow \mathcal{B}) \Leftrightarrow \mathcal{C}\ , \tag{A.50}$$

and, from Laws A.46 and A.47, it satisfies E.W. Dijkstra's *Golden Rule*:

$$\Rrightarrow \mathcal{A} \wedge \mathcal{B} \Leftrightarrow \mathcal{A} \Leftrightarrow \mathcal{B} \Leftrightarrow \mathcal{A} \vee \mathcal{B}\ . \tag{A.51}$$

A.2 Some predicate laws

In this section we consider laws concerning the universal and existential quantifiers, $\forall$ and $\exists$. Although for most practical purposes we wish the quantification to be *typed*

$$(\forall\, x : T \cdot \mathcal{A})$$
$$(\exists\, x : T \cdot \mathcal{A})\ ,$$

where T denotes a type and $\mathcal{A}$ is a formula, for simplicity we state our laws using untyped quantifications:

$$(\forall\, x \cdot \mathcal{A})$$
$$(\exists\, x \cdot \mathcal{A})\ .$$

Each can be converted to a law for typed quantification by uniform addition of type information, *provided the type is non-empty.* These laws enable us to convert between the two styles:

$$(\forall\, x : T \cdot \mathcal{A}) \;\equiv\; (\forall\, x \cdot x \in T \Rightarrow \mathcal{A}) \tag{A.52}$$

$$(\exists\, x : T \cdot \mathcal{A}) \;\equiv\; (\exists\, x \cdot x \in T \wedge \mathcal{A})\ , \tag{A.53}$$

where the simple formula $x \in T$ means 'x is in the set T'.

For more general constraints than typing, we have these abbreviations as well, which include a range formula $\mathcal{R}$:

$$(\forall\, x : T \mid \mathcal{R} \cdot \mathcal{A}) \;\equiv\; (\forall\, x \cdot x \in T \wedge \mathcal{R} \Rightarrow \mathcal{A}) \tag{A.54}$$

$$(\exists\, x : T \mid \mathcal{R} \cdot \mathcal{A}) \;\equiv\; (\exists\, x \cdot x \in T \wedge \mathcal{R} \wedge \mathcal{A})\ , \tag{A.55}$$

Note that A.52 and A.54 introduce implication, but A.53 and A.55 introduce conjunction.

A.2.1 Substitution

Recall (p.8) that we write substitution of a term E for a variable x in a formula $\mathcal{A}$ as

$$\mathcal{A}[x \backslash E] \ ,$$

and we write the multiple substitution of terms E and F for variables x and y respectively as

$$\mathcal{A}[x, y \backslash E, F] \ .$$

In simple cases, such substitutions just replace the variable by the term. In more complex cases, however, we must take account of whether variables are free or bound. Suppose, for example, that $\mathcal{A}$ is the formula $(\exists\, x \cdot x \neq y) \wedge x = y$; then

$$\begin{array}{rll} & \mathcal{A}[x \backslash y] & \text{is} \quad (\exists\, x \cdot x \neq y) \wedge y = y \ , \\ \text{but} & \mathcal{A}[y \backslash x] & \text{is} \quad (\exists\, z \cdot z \neq x) \wedge x = x \ . \end{array}$$

The variable z is *fresh*, not appearing in $\mathcal{A}$. In the first case, $x \neq y$ is unaffected because *that* occurrence of x is bound by $\exists\, x$. Indeed, since we could have used any other letter (except y) without affecting the meaning of the formula — and it would not have been replaced in that case — we do not replace it in this case either. The occurrence of x in $x = y$ is free, however, and the substitution occurs.

In the second case, since both occurrences of y are free, both are replaced by x. But on the left we must not 'accidentally' quantify over the newly introduced x — $(\exists\, x \cdot x \neq x)$ would be wrong — so we change (before the substitution) the bound x to a fresh variable z.

Finally, note that multiple substitution can differ from successive substitution:

$$\begin{array}{rll} & \mathcal{A}[y \backslash x][x \backslash y] & \text{is} \quad (\exists\, z \cdot z \neq y) \wedge y = y \\ \text{but} & \mathcal{A}[y, x \backslash x, y] & \text{is} \quad (\exists\, z \cdot z \neq x) \wedge y = x \ . \end{array}$$

A.2.2 The one-point laws

These laws allow quantifiers to be eliminated in many cases. They are called 'one-point' because the bound variable is constrained to take one value exactly. If x does not occur (free) in the term E, then

$$(\forall\, x \cdot x = E \Rightarrow \mathcal{A}) \ \equiv \mathcal{A}[x \backslash E] \equiv \ (\exists\, x \cdot x = E \wedge \mathcal{A}) \ . \tag{A.56}$$

If the type T in Laws A.52 and A.53 is finite, say $\{a, b\}$, we have the similar

$$(\forall\, x : \{a, b\} \cdot \mathcal{A}) \ \equiv \ \mathcal{A}[x \backslash a] \wedge \mathcal{A}[x \backslash b] \tag{A.57}$$

$$(\exists\, x : \{a, b\} \cdot \mathcal{A}) \ \equiv \ \mathcal{A}[x \backslash a] \vee \mathcal{A}[x \backslash b] \ . \tag{A.58}$$

Those can be extended to larger (but still finite) types $\{a, b, \cdots, z\}$. We are led to think, informally, of universal and existential quantification as infinite conjunction and disjunction respectively over all the constants of our logic:

$$\begin{array}{lll} (\forall\, x : \mathbb{N} \bullet \mathcal{A}) & \text{represents} & \mathcal{A}(0) \wedge \mathcal{A}(1) \cdots \\ (\exists\, x : \mathbb{N} \bullet \mathcal{A}) & \text{represents} & \mathcal{A}(0) \vee \mathcal{A}(1) \cdots \end{array}$$

A.2.3 Quantifiers alone

Quantification is idempotent:

$$(\forall\, x \bullet (\forall\, x \bullet \mathcal{A})) \;\equiv\; (\forall\, x \bullet \mathcal{A}) \tag{A.59}$$

$$(\exists\, x \bullet (\exists\, x \bullet \mathcal{A})) \;\equiv\; (\exists\, x \bullet \mathcal{A}) \;. \tag{A.60}$$

Extending de Morgan's laws A.18 and A.19, we have

$$\neg\,(\forall\, x \bullet \mathcal{A}) \;\equiv\; (\exists\, x \bullet \neg\mathcal{A}) \tag{A.61}$$

$$\neg\,(\exists\, x \bullet \mathcal{A}) \;\equiv\; (\forall\, x \bullet \neg\mathcal{A}) \;. \tag{A.62}$$

A.2.4 Extending the commutative laws

These laws extend the commutativity of $\wedge$ and $\vee$:

$$(\forall\, x \bullet (\forall\, y \bullet \mathcal{A})) \;\equiv (\forall\, x, y \bullet \mathcal{A}) \equiv\; (\forall\, y \bullet (\forall\, x \bullet \mathcal{A})) \tag{A.63}$$

$$(\exists\, x \bullet (\exists\, y \bullet \mathcal{A})) \;\equiv (\exists\, x, y \bullet \mathcal{A}) \equiv\; (\exists\, y \bullet (\exists\, x \bullet \mathcal{A})) \;. \tag{A.64}$$

A.2.5 Quantifiers accompanied

Extending the associative and previous laws,

$$(\forall\, x \bullet \mathcal{A} \wedge \mathcal{B}) \;\equiv\; (\forall\, x \bullet \mathcal{A}) \wedge (\forall\, x \bullet \mathcal{B}) \tag{A.65}$$

$$(\exists\, x \bullet \mathcal{A} \vee \mathcal{B}) \;\equiv\; (\exists\, x \bullet \mathcal{A}) \vee (\exists\, x \bullet \mathcal{B}) \tag{A.66}$$

$$(\exists\, x \bullet \mathcal{A} \Rightarrow \mathcal{B}) \;\equiv\; (\forall\, x \bullet \mathcal{A}) \Rightarrow (\exists\, x \bullet \mathcal{B}) \;. \tag{A.67}$$

Here are weaker laws (using $\Rrightarrow$ rather than $\equiv$) which are nonetheless useful:

$$(\forall\, x \bullet \mathcal{A}) \;\Rrightarrow\; (\exists\, x \bullet \mathcal{A}) \tag{A.68}$$

$$(\forall\, x \bullet \mathcal{A}) \vee (\forall\, x \bullet \mathcal{B}) \;\Rrightarrow\; (\forall\, x \bullet \mathcal{A} \vee \mathcal{B}) \tag{A.69}$$

$$(\forall\, x \bullet \mathcal{A} \Rightarrow \mathcal{B}) \;\Rrightarrow\; (\forall\, x \bullet \mathcal{A}) \Rightarrow (\forall\, x \bullet \mathcal{B}) \tag{A.70}$$

$$(\exists\, x \bullet \mathcal{A} \wedge \mathcal{B}) \;\Rrightarrow\; (\exists\, x \bullet \mathcal{A}) \wedge (\exists\, x \bullet \mathcal{B}) \tag{A.71}$$

$$(\exists\, x \bullet \mathcal{A}) \Rightarrow (\exists\, x \bullet \mathcal{B}) \;\Rrightarrow\; (\exists\, x \bullet \mathcal{A} \Rightarrow \mathcal{B}) \tag{A.72}$$

$$(\exists\, y \bullet (\forall\, x \bullet \mathcal{A})) \;\Rrightarrow\; (\forall\, x \bullet (\exists\, y \bullet \mathcal{A})) \;. \tag{A.73}$$

A.2.6 Manipulation of quantifiers

If a variable has no free occurrences, its quantification is superfluous:

$$(\forall\, x \bullet \mathcal{A}) \;\equiv \mathcal{A} \quad \text{if } x \text{ is not free in } \mathcal{A} \tag{A.74}$$
$$(\exists\, x \bullet \mathcal{A}) \;\equiv \mathcal{A} \quad \text{if } x \text{ is not free in } \mathcal{A}\ . \tag{A.75}$$

Other useful laws of this kind are the following, many of which are specialisations of laws A.65 to A.67. In each case, x must not be free in the formula $\mathcal{N}$:

$$(\forall\, x \bullet \mathcal{N} \wedge \mathcal{B}) \quad\equiv\quad \mathcal{N} \wedge (\forall\, x \bullet \mathcal{B}) \tag{A.76}$$
$$(\forall\, x \bullet \mathcal{N} \vee \mathcal{B}) \quad\equiv\quad \mathcal{N} \vee (\forall\, x \bullet \mathcal{B}) \tag{A.77}$$
$$(\forall\, x \bullet \mathcal{N} \Rightarrow \mathcal{B}) \quad\equiv\quad \mathcal{N} \Rightarrow (\forall\, x \bullet \mathcal{B}) \tag{A.78}$$
$$(\forall\, x \bullet \mathcal{A} \Rightarrow \mathcal{N}) \quad\equiv\quad (\exists\, x \bullet \mathcal{A}) \Rightarrow \mathcal{N} \tag{A.79}$$
$$(\exists\, x \bullet \mathcal{N} \wedge \mathcal{B}) \quad\equiv\quad \mathcal{N} \wedge (\exists\, x \bullet \mathcal{B}) \tag{A.80}$$
$$(\exists\, x \bullet \mathcal{N} \vee \mathcal{B}) \quad\equiv\quad \mathcal{N} \vee (\exists\, x \bullet \mathcal{B}) \tag{A.81}$$
$$(\exists\, x \bullet \mathcal{N} \Rightarrow \mathcal{B}) \quad\equiv\quad \mathcal{N} \Rightarrow (\exists\, x \bullet \mathcal{B}) \tag{A.82}$$
$$(\exists\, x \bullet \mathcal{A} \Rightarrow \mathcal{N}) \quad\equiv\quad (\forall\, x \bullet \mathcal{A}) \Rightarrow \mathcal{N}\ . \tag{A.83}$$

Bound variables can be renamed, as long as the new name does not conflict with existing names:

$$(\forall\, x \bullet \mathcal{A}) \equiv \;\; (\forall\, y \bullet \mathcal{A}[x \backslash y]) \quad \text{if } y \text{ is not free in } \mathcal{A} \tag{A.84}$$
$$(\exists\, x \bullet \mathcal{A}) \equiv \;\; (\exists\, y \bullet \mathcal{A}[x \backslash y]) \quad \text{if } y \text{ is not free in } \mathcal{A}\ . \tag{A.85}$$

Finally, we have for any term E,

$$(\forall\, x \bullet \mathcal{A}) \;\Rrightarrow\; \mathcal{A}[x \backslash E] \tag{A.86}$$
$$\mathcal{A}[x \backslash E] \;\Rrightarrow\; (\exists\, x \bullet \mathcal{A})\ . \tag{A.87}$$

If A is true for all x, then it is true for E in particular; and if A is true for E, then certainly it is true for some x.