Prefixing and choice

$$P \mathbin{\square} P = P \qquad \langle\square\text{-idem}\rangle \qquad (1.1)$$

$$P \sqcap P = P \qquad \langle\sqcap\text{-idem}\rangle \qquad (1.2)$$

$$P \mathbin{\square} Q = Q \mathbin{\square} P \qquad \langle\square\text{-sym}\rangle \qquad (1.3)$$

$$P \sqcap Q = Q \sqcap P \qquad \langle\sqcap\text{-sym}\rangle \qquad (1.4)$$

$$P \mathbin{\square} (Q \mathbin{\square} R) = (P \mathbin{\square} Q) \mathbin{\square} R \qquad \langle\square\text{-assoc}\rangle \qquad (1.5)$$

$$P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap R \qquad \langle\sqcap\text{-assoc}\rangle \qquad (1.6)$$

Prefixing and choice II

$$P \Box (Q \sqcap R) = (P \Box Q) \sqcap (P \Box R) \qquad \langle\Box\text{-dist}\rangle \quad (1.7)$$

$$P \Box \bigsqcap S = \bigsqcap\{P \Box Q \mid Q \in S\} \qquad \langle\Box\text{-Dist}\rangle \quad (1.8)$$

$$a \to (P \sqcap Q) = (a \to P) \sqcap (a \to Q) \qquad \langle\text{prefix-dist}\rangle \quad (1.9)$$

$$a \to \bigsqcap S = \bigsqcap\{a \to Q \mid Q \in S\} \qquad \langle\text{prefix-Dist}\rangle \quad (1.10)$$

$$?x : A \to (P \sqcap Q) = (?x : A \to P) \sqcap (?x : A \to Q) \qquad \langle\text{input-dist}\rangle \quad (1.11)$$

$$?x : A \to \bigsqcap S = \bigsqcap\{?x : A \to Q \mid Q \in S\} \qquad \langle\text{input-Dist}\rangle \quad (1.12)$$

$$P \sqcap (Q \Box R) = (P \sqcap Q) \Box (P \sqcap R) \qquad \langle\sqcap\text{-}\Box\text{-dist}\rangle \quad (1.13)$$

Prefixing and choice III

$$
\begin{aligned}
(?x : A \rightarrow P) \,\Box\, (?x : B \rightarrow Q) \;=\; & \\
?x : A \cup B \rightarrow \; & ((P \sqcap Q) \\
& \lessdot x \in A \cap B \gtrdot \\
& (P \lessdot x \in A \gtrdot Q))
\end{aligned}
\qquad \langle \Box\text{-step} \rangle \quad (1.14)
$$

$$STOP \;= ?x : \{\} \rightarrow P \qquad \langle STOP\text{-step} \rangle \quad (1.15)$$

$$STOP \,\Box\, P \;=\; P \qquad \langle \Box\text{-unit} \rangle \quad (1.16)$$

Conditional choice

$$P \lhd b \rhd P \;=\; P \qquad \langle \lhd \cdot \rhd\text{-idem} \rangle \quad (1.17)$$

$$(P \sqcap Q) \lhd b \rhd R \;=\; (P \lhd b \rhd R) \sqcap (Q \lhd b \rhd R) \qquad \langle \lhd \cdot \rhd\text{-dist-l} \rangle \quad (1.18)$$

$$R \lhd b \rhd (P \sqcap Q) \;=\; (R \lhd b \rhd P) \sqcap (R \lhd b \rhd Q) \qquad \langle \lhd \cdot \rhd\text{-dist-r} \rangle \quad (1.19)$$

$$P \lhd true \rhd Q \;=\; P \qquad \langle \lhd \text{true} \rhd\text{-id} \rangle \quad (1.20)$$

$$P \lhd false \rhd Q \;=\; Q \qquad \langle \lhd \text{false} \rhd\text{-id} \rangle \quad (1.21)$$

$$P \,\Box\, (Q \lhd b \rhd R) \;=\; (P \,\Box\, Q) \lhd b \rhd (P \,\Box\, R) \qquad \langle \lhd \cdot \rhd\text{-}\Box\text{-dist} \rangle \quad (1.22)$$

$$\mu\, p.P \;=\; P[\mu\, p.P / p] \qquad \langle \mu\text{-unwind} \rangle \quad (1.23)$$

Parallel step laws

Synchronous parallel:

$$?x : A \to P \parallel ?x : B \to Q \;=\; ?x : A \cap B \to (P \parallel Q) \qquad \langle\parallel\text{-step}\rangle \qquad (2.1)$$

Alphabetised parallel: $P = ?x : A \to P'$ and $Q = ?x : B \to Q'$ and

$$C = (A \cap (X \setminus Y)) \cup (B \cap (Y \setminus X)) \cup (A \cap B \cap X \cap Y)$$

implies

$$P \;{}_X\!\parallel_Y\; Q \;=\; ?x : C \to (\; P' \lessdot x \in X \gtrdot P \;{}_X\!\parallel_Y\; Q' \lessdot x \in Y \gtrdot Q) \qquad \langle{}_X\!\parallel_Y\text{-step}\rangle \qquad (2.2)$$

Parallel

$$P \ {}_X\|_Y \ (Q \sqcap R) \ = \ (P \ {}_X\|_Y \ Q) \sqcap (P \ {}_X\|_Y \ R) \qquad \langle {}_X\|_Y\text{-dist}\rangle \qquad (2.3)$$

$$P \ {}_X\|_Y \ Q \ = \ Q \ {}_Y\|_X \ P \qquad \langle {}_X\|_Y\text{-sym}\rangle \qquad (2.4)$$

$$(P \ {}_X\|_Y \ Q) \ {}_{X\cup Y}\|_Z \ R \ = \ P \ {}_X\|_{Y\cup Z} \ (Q \ {}_Y\|_Z \ R) \qquad \langle {}_X\|_Y\text{-assoc}\rangle \qquad (2.5)$$

Interleaving

If $P = ?x : A \to P'$ and $Q = ?x : B \to Q'$ then

$$P \mathbin{|||} Q = ?x : A \cup B \to \begin{array}{l}(P' \mathbin{|||} Q) \sqcap (P \mathbin{|||} Q') \\ \lessdot x \in A \cap B \gtrdot \\ (P' \mathbin{|||} Q) \lessdot x \in A \gtrdot (P \mathbin{|||} Q')\end{array} \qquad \langle|||\text{-step}\rangle \quad (2.6)$$

$$P \mathbin{|||} Q = P \mathbin{|||} Q \qquad \langle|||\text{-sym}\rangle \quad (2.7)$$

$$(P \mathbin{|||} Q) \mathbin{|||} R = P \mathbin{|||} (Q \mathbin{|||} R) \qquad \langle|||\text{-assoc}\rangle \quad (2.8)$$

$$P \mathbin{|||} (Q \sqcap R) = (P \mathbin{|||} Q) \sqcap (P \mathbin{|||} R) \qquad \langle|||\text{-dist}\rangle \quad (2.9)$$

Generalised parallel

If $P = ?x : A \to P'$ and $Q = ?x : B \to Q'$, then

$$
\begin{aligned}
P \underset{X}{\|} Q \;=\; ?x : C \to \;\; & (P' \| Q') \lessdot x \in X \gtrdot \\
& (((P' \underset{X}{\|} Q) \sqcap (P \underset{X}{\|} Q')) \lessdot x \in A \cap B \gtrdot \\
& ((P' \underset{X}{\|} Q) \lessdot x \in A \gtrdot (P \underset{X}{\|} Q')))
\end{aligned}
$$

$\langle \underset{X}{\|}$-step$\rangle$ (2.10)

$$P \underset{X}{\|} Q \;=\; Q \underset{X}{\|} P$$

$\langle \underset{X}{\|}$-sym$\rangle$ (2.11)

$$P \underset{X}{\|} (Q \sqcap R) \;=\; (P \underset{X}{\|} Q) \sqcap (P \underset{X}{\|} R)$$

$\langle \underset{X}{\|}$-dist$\rangle$ (2.12)

$$P \underset{X}{\|} (Q \underset{X}{\|} R) \;=\; (P \underset{X}{\|} Q) \underset{X}{\|} R)$$

$\langle \underset{X}{\|}$-assoc$\rangle$ (2.13)

Hiding

$$(P \sqcap Q) \setminus X = (P \setminus X) \sqcap (Q \setminus X) \qquad \langle\text{hide-dist}\rangle \tag{3.1}$$

$$(P \setminus Y) \setminus X = (P \setminus X) \setminus Y \qquad \langle\text{hide-sym}\rangle \tag{3.2}$$

$$(P \setminus Y) \setminus X = P \setminus (X \cup Y) \qquad \langle\text{hide-combine}\rangle \tag{3.3}$$

$$P \setminus \{\} = P \qquad \langle\text{null hiding}\rangle \tag{3.4}$$

$$(a \to P) \setminus X = \begin{cases} P \setminus X & \text{if } a \in X \\ a \to (P \setminus X) & \text{if } a \notin X \end{cases} \qquad \langle\text{hide-step 1}\rangle \tag{3.5}$$

Hiding II

$$(?x : A \to P) \setminus X \;=\; \begin{cases} ?x : A \to (P \setminus X) & \text{if } A \cap X = \{\} \\ (?x : (A \smallsetminus X) \to (P \setminus X)) \\ \rhd \bigsqcap\{(P[a/x]) \setminus X \mid a \in A \cap X\} & \text{if } A \cap X \neq \{\} \end{cases} \qquad \langle\text{hide-step}\rangle \qquad (3.6)$$

$$(P \;{}_X\|_Y\; Q) \setminus Z \;=\; (P \setminus Z \cap X) \;{}_X\|_Y\; (Q \setminus Z \cap Y) \quad \text{provided } X \cap Y \cap Z = \{\} \qquad \langle\text{hide-}{}_X\|_Y\text{-dist}\rangle \qquad (3.7)$$

$$(P \underset{X}{\|} Q) \setminus Z \;=\; (P \setminus Z) \underset{X}{\|} (Q \setminus Z) \quad \text{provided } X \cap Z = \{\} \qquad \langle\text{hide-}\underset{X}{\|}\text{-dist}\rangle \qquad (3.8)$$

Renaming

$$f[P \underset{X}{\|} Q] = f[P] \underset{f(X)}{\|} f[Q] \quad \text{if } f \text{ is 1–1} \qquad \langle f[\cdot]\text{-}\underset{X}{\|}\text{-dist}\rangle \qquad (3.9)$$

$$f[P \;{}_X\|_Y\; Q] = f[P] \;{}_{f(X)}\|_{f(Y)}\; f[Q] \quad \text{if } f \text{ is 1–1} \qquad \langle f[\cdot]\text{-}{}_X\|_Y\text{-dist}\rangle \qquad (3.10)$$

$$f[P \setminus X] = f[P] \setminus f(X) \quad \text{if } f \text{ is 1–1} \qquad \langle f[\cdot]\text{-hide-sym}\rangle \qquad (3.11)$$

$$f[P \setminus X] = P \setminus X \quad \text{if } f(y) = y \text{ for all } y \in \Sigma \setminus X \qquad \langle f[\cdot]\text{-hide-null}\rangle \qquad (3.12)$$

$$(P \sqcap Q)[\![R]\!] = P[\![R]\!] \sqcap Q[\![R]\!] \qquad \langle [\![R]\!]\text{-dist}\rangle \qquad (3.13)$$

$$(P \,\square\, Q)[\![R]\!] = P[\![R]\!] \,\square\, Q[\![R]\!] \qquad \langle [\![R]\!]\text{-}\square\text{-dist}\rangle \qquad (3.14)$$

Renaming II

$$(?x : A \to P)[\![R]\!] \;= ?y : R(A) \to \bigsqcap\{(P[z/x])[\![R]\!] \mid z \in A \wedge z\; R\; y\} \quad \langle [\![R]\!]\text{-step}\rangle \quad (3.15)$$

$$(P[\![R]\!])[\![R']\!] \;=\; P[\![R \circ R']\!] \quad \langle [\![R]\!]\text{-combine}\rangle \quad (3.16)$$

$$f[g[P]] \;=\; (f \circ g)[P] \quad \langle f[\cdot]\text{-combine}\rangle \quad (3.17)$$

Piping

$$(P \sqcap Q) \gg R \;=\; (P \gg R) \sqcap (Q \gg R) \qquad \langle\gg\text{-dist-l}\rangle \tag{4.1}$$

$$P \gg (Q \sqcap R) \;=\; (P \gg Q) \sqcap (P \gg R) \qquad \langle\gg\text{-dist-r}\rangle \tag{4.2}$$

$$(P \gg Q) \gg R \;=\; P \gg (Q \gg R) \qquad \langle\gg\text{-assoc}\rangle \tag{4.3}$$

$$\mathit{left}?x \to P \gg \mathit{left}?y \to Q \;=\; \mathit{left}?x \to (P \gg \mathit{left}?y \to Q) \qquad \langle\gg\text{-step 1}\rangle \tag{4.4}$$

$$\mathit{right}!x \to P \gg \mathit{right}!y \to Q \;=\; \mathit{right}!y \to (\mathit{right}!x \to P \gg Q) \qquad \langle\gg\text{-step 2}\rangle \tag{4.5}$$

$$\mathit{right}!x \to P \gg \mathit{left}?y \to Q \;=\; P \gg Q[x/y] \qquad \langle\gg\text{-step 3}\rangle \tag{4.6}$$

$$\mathit{left}?x \to P \gg \mathit{right}!y \to Q \;=\; \begin{array}{l} \mathit{left}?x \to (P \gg \mathit{right}!y \to Q) \\ \Box\;\; \mathit{right}!y \to (\mathit{left}?x \to P \gg Q) \end{array} \qquad \langle\gg\text{-step 4}\rangle \tag{4.7}$$

Piping and Enslavement

Let

$$\begin{array}{rcl} P & = & (\mathit{left}?x : A \to P') \,\Box\, (\mathit{right}?x : B \to P'') \\ Q & = & (\mathit{left}?y : C \to Q') \,\Box\, (\mathit{right}?y : D \to Q'') \end{array}$$

$$\begin{array}{rl} P \gg Q = & (\mathit{left}?x : A \to (P' \gg Q)) \,\Box\, (\mathit{right}?y : D \to (P \gg Q'')) \\ & \quad \lessdot B \cap C = \{\} \gtrdot \\ & (\mathit{left}?x : A \to (P' \gg Q)) \,\Box\, (\mathit{right}?y : D \to (P \gg Q'') \\ & \rhd \ \sqcap\{P''[z/x] \gg Q'[z/y] \mid z \in B \cap C\} \end{array} \qquad \langle \gg\text{-step} \rangle \qquad (4.8)$$

The only quoted law of enslavement (though there are many more) is

$$(P /\!\!/_Y Q) /\!\!/_Z R = (P /\!\!/_Z R) /\!\!/_Y Q \qquad \langle /\!\!/\text{-sym} \rangle \qquad (4.9)$$

Sequencing and $SKIP$

$$(P \sqcap Q);\ R = (P;\ R) \sqcap (Q;\ R) \qquad \langle ;\text{-dist-l}\rangle \qquad (6.1)$$

$$P;\ (Q \sqcap R) = (P;\ Q) \sqcap (P;\ R) \qquad \langle ;\text{-dist-r}\rangle \qquad (6.2)$$

$$P;\ (Q;\ R) = (P;\ Q);\ R \qquad \langle ;\text{-assoc}\rangle \qquad (6.3)$$

$$SKIP;\ P = P \qquad \langle ;\text{-unit-l}\rangle \qquad (6.4)$$

$$P;\ SKIP = P \qquad \langle ;\text{-unit-r}\rangle \qquad (6.5)$$

$$P \;\Box\; SKIP = P \rhd SKIP \qquad \langle \Box\text{-}SKIP \text{ resolve}\rangle \qquad (6.6)$$

Sequencing and $SKIP$ II

Provided x is not free in Q:

$$(?x : A \to P);\ Q\ = ?x : A \to (P;\ Q) \qquad \langle ;\text{-step}\rangle \qquad (6.7)$$

$$((?x : A \to P) \triangleright SKIP);\ Q\ =\ (?x : A \to (P;\ Q)) \triangleright Q \qquad \langle SKIP\text{-}; \text{-step}\rangle \qquad (6.8)$$

$$SKIP \setminus X\ =\ SKIP \qquad \langle SKIP\text{-hide-Id}\rangle \qquad (6.9)$$

$$SKIP[\![R]\!]\ =\ SKIP \qquad \langle SKIP\text{-}[\![R]\!]\text{-Id}\rangle \qquad (6.10)$$

Distributed termination

$$SKIP \;{}_X\|_Y\; SKIP = SKIP \qquad \langle {}_X\|_Y\text{-termination}\rangle \quad (6.11)$$

$$SKIP \underset{X}{\|} SKIP = SKIP \qquad \langle \underset{X}{\|}\text{-termination}\rangle \quad (6.12)$$

$$SKIP \gg SKIP = SKIP \qquad \langle \gg\text{-termination}\rangle \quad (6.13)$$

$$SKIP /\!\!/_X SKIP = SKIP \qquad \langle /\!\!/\text{-termination}\rangle \quad (6.14)$$

$$SKIP \underset{X}{\|} (?x : A \to P) = ?x : A \setminus X \to (SKIP \underset{X}{\|} P) \qquad \langle \underset{X}{\|}\text{-preterm}\rangle \quad (6.15)$$

$$SKIP \;|||\; P = P \qquad \langle |||\text{-unit}\rangle \quad (6.16)$$

α-conversion and zero laws

$$?x : A \to P = ?y : A \to P[y/x] \quad \text{if } y \text{ is not free in } P \qquad \langle\text{input } \alpha\text{-cnv}\rangle \quad (11.1)$$

$$\mathbf{div} \sqcap P = \mathbf{div} \qquad \langle\sqcap\text{-zero}^{\mathcal{N}}\rangle \quad (11.2)$$

$$\mathbf{div} \,\Box\, P = \mathbf{div} \qquad \langle\Box\text{-zero}^{\mathcal{N}}\rangle \quad (11.3)$$

$$\mathbf{div} \underset{X}{\|} P = \mathbf{div} \qquad \langle\underset{X}{\|}\text{-zero}^{\mathcal{N}}\rangle \quad (11.4)$$

$$\mathbf{div} \;{}_X\|_Y\; P = \mathbf{div} \qquad \langle{}_X\|_Y\text{-zero}^{\mathcal{N}}\rangle \quad (11.5)$$

$$\mathbf{div} \;|||\; P = \mathbf{div} \qquad \langle|||\text{-zero}^{\mathcal{N}}\rangle \quad (11.6)$$

$$\mathbf{div} \setminus X = \mathbf{div} \qquad \langle\text{hide-zero}\rangle \quad (11.7)$$

$$\mathbf{div}[\![R]\!] = \mathbf{div} \qquad \langle[\![R]\!]\text{-zero}\rangle \quad (11.8)$$

Miscellaneous laws for normalisation

$$
\begin{array}{l}
((?x : A \to P) \,\Box\, SKIP) \setminus X \quad = \\
\qquad (?x : A \to (P \setminus X)) \,\Box\, SKIP \\
\qquad \not< A \cap X = \{\} \not> \\
\qquad ((?x : A \setminus X \to (P \setminus X)) \,\Box\, SKIP \\
\qquad \sqcap \bigsqcap \{P[a/x] \setminus X \mid a \in A \cap X\}
\end{array}
$$

$\langle$hide-$SKIP$-step$\rangle$ (11.9)

$$
\begin{array}{l}
((?x : A \to P) \,\Box\, SKIP) \underset{X}{\|} SKIP \quad = \\
\qquad (?x : A \setminus X \to (P \underset{X}{\|} SKIP)) \,\Box\, SKIP
\end{array}
$$

$\langle \underset{X}{\|}$-$\Box\, SKIP\rangle$ (11.10)

More miscellaneous laws for normalisation

$$
\begin{aligned}
&(P \rhd Q) \underset{X}{\|} (?x : A \to R)) \;\; = \\
&\qquad ?x : A \setminus X \to ((P \rhd Q) \underset{X}{\|} R) \\
&\qquad \Box\, ((P \underset{X}{\|} ?x : A \to R) \rhd (Q \underset{X}{\|} ?x : A \to R)) \\
&\qquad \text{if } x \text{ is not free in } P \text{ or } Q
\end{aligned}
\qquad \langle \underset{X}{\|}\text{-}\rhd\text{-input} \rangle \;\; (11.11)
$$

$$
\begin{aligned}
&(P \rhd Q) \underset{X}{\|} (R \rhd S) \;\; = \\
&\qquad (P \underset{X}{\|} R) \rhd ((Q \underset{X}{\|} (R \rhd S)) \sqcap ((P \rhd Q) \underset{X}{\|} S))
\end{aligned}
\qquad \langle \underset{X}{\|}\text{-}\rhd\text{-split} \rangle \;\; (11.12)
$$

Laws for $\mathcal{F}$ and $\mathcal{T}$

$$P \sqcap \mathbf{div} \;=\; P \qquad \langle \sqcap\text{-unit}^{\mathcal{T},\mathcal{F}} \rangle \quad (11.13)$$

$$P \;\Box\; Q \;=\; P \sqcap Q \qquad \langle \text{trace-equiv}^{\mathcal{T}} \rangle \quad (11.14)$$